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INTEGRABILITY AND NONINTEGRABILITY OF DYNAMICAL SYSTEMS

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Preface

This book is devoted to the integrability and nonintegrability of systems of nonlinear differential equations. Differential equations and dynamical systems appear naturally in the description of many phenomena for which local processes are known. For instance, most physical laws, such as conservation of mass, energy, and momentum are *local* laws. The central problem is then to obtain *global* information on these phenomena. These elementary processes are typically nonlinear and, assuming continuity of the states of the system (the dependent variables) in time and space (the independent variables), their evolution is governed by *nonlinear* differential equations. For example, in classical physics, the gravitational forces between masses is nonlinear as are the electromagnetic interactions. In hydrodynamics, the nonlinearity of the Navier-Stokes equation comes from inertial effects. Also, autocatalytic chemical reactions are described by nonlinear differential equations through the mass-action law. These nonlinear effects give rise to complex structures whose complete description can be extremely difficult. Once the local equations are formulated in a particular context, the next problem is to "solve" these equations. Already, in this simple statement, there is an ambiguity. For the physicist, the applied mathematician or the chemist, to "solve" an equation means to obtain global information on the solution and, if possible, derive a closed-form solution for which the state of the dependent variables may be predicted for all given independent variables. In this sense, an equation can be solved if it can be locally represented by known functions. The mathematician, however, is often interested in a more fundamental problem related to the existence and uniqueness of the solutions, a prerequisite of any subsequent analytical approach.

The first attempt to solve differential equations either explicitly or by series expansions goes back to Euler, Newton, and Leibniz. The theory of integration for the equations of motion was subsequently expanded by the work of the analysts and mechanicians associated with the names of Lagrange, Poisson, Hamilton, and Liouville in the late 18th and 19th centuries. The basic idea underlying these works is that the solution can always be represented by the combination of known functions or by perturbation expansions. The notion of "integrability" was then introduced to describe the property of equations for which all local and global information can be obtained either explicitly from the solutions or implicitly from the constants of the motion.

In a reductionist approach, we can single out two different works that have radically changed the program of classical mechanics of the 19th century, and will serve as guidelines throughout this book. The first of these works is Kovalevskaya's study of the Euler equations for the motion of a rigid body with a fixed point. She used an ingenuous and innovative technique based on the behavior of the solution near the singularities in the complex plane to show that apart from the known integrable cases and a new one that she discovered, there is no other case for which the solution can be expressed exactly in terms of single-valued functions. In essence, she proved that within the class of single-valued functions, the general Euler equations are not integrable.

Second, Poincaré studied the existence of constants of motion for integrable Hamiltonians under small perturbations. He showed that, in general, there is no additional constant of motion other than the Hamiltonian itself which is analytic in the expansion parameter. That is, if a constant of motion exists for the unperturbed Hamiltonian, then, it cannot exist continuously as the perturbation parameter is increased. In his study of celestial mechanics, Poincaré also developed a geometric theory of solutions. His idea was to study asymptotic solutions as geometric sets which define the global qualitative behavior of solutions in the long time limit. Poincaré introduced the concept of homoclinic and heteroclinic orbits which connect fixed points to themselves and showed that perturbations of these orbits are the source of complex behaviors. He noticed that if the three-body problem could be solved, "the transcendents needed to solve it would differ radically form all the known ones" ("les transcendantes qu'il faudrait imaginer pour le résoudre diffèrent de toutes celles que nous connaissons") (Poincaré, 1899, p. 391). In many respects, Poincaré's investigations of complex motion were several decades ahead of their time. Indeed, his study was based on an entirely different approach, the analysis of the *qualitative* behavior of solutions. To determine the global behavior of the solution in the long-time limit, his idea was to exploit the topological properties of the solutions in phase space together with the analytical properties of the equation

Despite their differences, the approaches of Kovalevskaya and Poincaré share a common feature. The local analysis of the differential equation, close to its complex time singularities for Kovalevskaya and its phase space singularities for Poincaré, allows us to find global properties of the system. As a consequence of these works, both mathematicians and physicists shifted their interests away from the theory of integrability. Mathematicians realized that the essence of the qualitative theory of differential equations was based on the notion of *dynamical systems* for which the abstract formulation was laid down by Birkhoff. Physicists did not fully appreciate the importance of nonlinearities until the 1960's. With the seminal works of Lorenz (Lorenz, 1963), on the numerical evidence of chaotic motion, and of Hénon and Heiles, on a nonintegrable two-degrees of freedom Hamiltonian (Hénon & Heiles, 1964), dynamical systems theory radically changed the way scientists think about nonlinear problems.

The success of dynamical systems theory was so overwhelming that exact methods for integration were considered for years useless and non-generic. In particular, chaos theory shadowed the equally important discovery by Zabusky and Kruskal (1965) of solitons for the Korteweg-de Vries equation. Solitons are elementary solutions of partial differential equations that have simple interaction laws reminiscent of linear systems and they are considered as the hallmark of integrability in nonlinear partial differential equations. Soon after, many other integrable systems were discovered or rediscovered. Strikingly, for years, chaos, strange attractors, and ergodicity have been considered as the important features of dynamical systems with few degrees of freedom. However, solitons, pattern formation and ordered structures were the key features of systems with infinite degrees of freedom modeled by nonlinear partial differential equations. These conceptual differences emerging from nonlinear models seem shocking and show how crucial is the understanding of the phenomena of integrability and nonintegrability in dynamical systems.

To further define the problem, we distinguish "solvability" from "integrability". "Integrability" is an intrinsic property of a given system imposing strong constraints on the way solutions evolve in phase space whereas "solvability" is related to the existence of closed-form solutions. However, a universal definition of integrability for dynamical systems seems elusive. Clearly, it should at least be compatible with the intuitive notion of regular or irregular behaviors. For dynamical systems, irregular behavior is usually associated with bounded dynamics sensitive to initial conditions with neighboring trajectories diverging locally in phase space with local exponential rates measured by the Lyapunov exponents. These exponents cannot be computed in general and their numerical evaluation, involving long-time averages, may be an extremely arduous task. Therefore, integrability cannot be simply defined by the lack of irregular behavior since many nonintegrable systems have regular dynamics. Our modest standpoint for the understanding of the problem of integrability and nonintegrability in dynamical system is singularity analysis; that is, the analysis of differential equations in complex time. We will consider several definitions of integrability and show how they relate to the structure of the solution viewed as functions of a complex variable. Then, we will develop simple algorithmic methods to detect systems which lack the fundamental properties of integrability. With further assumptions on nonintegrable systems, we will explicitly relate nonintegrability to the existence of irregular behaviors and show that seemingly contradictory aspects of nonlinear systems can be understood within singularity analysis.

What this book is not about

Integrability and dynamical systems have become such important theories that they have acquired over the years different meanings for different people. In writing this book, unless necessary, I have tried to include subjects which were not already covered by other textbooks. I felt that I could not improve on the excellent exposition or more talented writers. To avoid any confusion, I would like to give a partial list of subjects (all of great interests) usually associated with the notion of integrability that are not (or only partially) covered here: Integrable systems (Perelomov, 1990; Audin, 1996), Hamiltonian systems (Goldstein, 1980; Marsden & Ratiu, 1994; Kozlov, 1998), Discrete systems (Grammaticos *et al.*, 1999), Soliton theory and Inverse scattering techniques (Ablowitz & Segur, 1981; Newell, 1985; Ablowitz & Clarkson, 1991), Lie group analysis (Olver, 1993; Ibragimov, 1999), Dynamical systems and Chaos (Guckenheimer & Holmes, 1983; Wiggins, 1988; Perko, 1996).

What this book is about

This is mainly a book of concepts and methods. I hope that by carefully defining some general concepts and by showing many illustrative examples, I have given the reader the tools to tackle her or his problem. I have tried to adopt the view that studying the integrability or nonintegrability of a given dynamical system is not Black Magic and that, in fact, a systematic approach can be followed. To many, integrability is a mysterious notion that appears to occur seemingly randomly in the study of dynamical systems. True enough, integrability is rare, but we should cherish and fully understand these rare instances. Ultimately, they are the key to a thorough understanding of regular and irregular behavior of dynamical systems.

In Chapter 1, I consider two simple dynamical systems and try to formulate simple questions on the behaviors of their solutions which can be answered by considering the integrability of the systems. Chapter 2 is a general introduction to vector fields and first integrals. I define different notions of integrability based on the existence of first integrals and explore elementary properties of Lax pairs. Chapter 3 is dedicated to singularity analysis and the Painlevé property. Integrability is defined through the behavior of solutions in complex time. Chapter 4 further explores the algebraic and analytic properties of a large class of vector fields for which explicit matrix representations can be given. In Chapter 5, I use the local properties of vector fields in both phase space and complex time to develop methods to prove the nonintegrability of a given system. Chapter 6 presents an elementary introduction to the theory of integrable and nonintegrable Hamiltonian systems. The results developed in Chapters 3 and 4 are adapted to the special nature of Hamiltonians. In Chapter 7, I consider integrable systems under perturbations and study the effect of the perturbation, both in complex time and in phase space.

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Contents

Chapter 1:

Introduction

"Data aequatione quotcumque fluenes quantitates involvente,
fluxiones invenire et vice versa."[1]
Newton

In this chapter, we begin with examples of modest dynamical systems in low dimensions, describe some elementary properties of their solutions, and try to formulate key questions based on these observations. The answers to these questions will be addressed throughout this book.

1.1 A planar system

We start with a two-dimensional system of ordinary differential equations in the plane, studied in connection with the problem of the center (Lunkevich & Sibirskii, 1982; Delshams & Mir, 1997).

$$\dot{x} = y + 2\alpha xy \tag{1.1.a}$$
$$\dot{y} = -x + \gamma x^2 - \beta y^2 \tag{1.1.b}$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dx}{dt}$, and α, β, γ are arbitrary parameters. Alternatively, we rewrite this system as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where the vectors $\mathbf{x}$ and $\mathbf{f}(\mathbf{x})$ are given by $\mathbf{x} = (x, y)$ and $\mathbf{f} = (y + 2\alpha xy, -x + \gamma x^2 - \beta y^2)$, respectively.[2]

The central problem is to determine some properties of this system. Does it have fixed points, periodic orbits, limit cycles? Are the solutions bounded? Can they be expressed in finite terms? In short, can we say anything at all about the solution of these equations?

1.1.1 A dynamical system approach

The basic tool for the analysis of such problems is the theory of dynamical systems (see for instance (Guckenheimer & Holmes, 1983; Wiggins, 1988; Perko, 1996)). We assume that the reader has some basic knowledge of this theory in order to proceed with the following analysis. As a first step, we find the *fixed points*, that is, the constant solutions of the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. In our case, there are

[1] In a letter to Oldenburg (forwarded to Leibniz) of October 24, 1676, Newton defines the notion of differential equations: *Given any equation, involving fluent quantities (integrals), to find the fluxions (differentials), and vice versa.*

[2] Note that by a rescaling, we can obtain an equivalent system with two free parameters. However, we leave the system as it is written in order to discuss the solutions in full generality.

at most four such solutions located at:

$$\mathbf{x}^{(1)} = (0,0), \tag{1.2.a}$$

$$\mathbf{x}^{(2)} = (0,\frac{1}{\gamma}), \tag{1.2.b}$$

$$\mathbf{x}^{(3,4)} = (-\frac{1}{2\alpha},\frac{\pm 1}{2\alpha}\sqrt{\frac{\gamma+2\alpha}{\beta}}). \tag{1.2.c}$$

Once identified, a local analysis around each fixed point can be carried out by computing the *linear eigenvalues* which are the exponents of the linear solutions for the system obtained locally around each fixed point. That is, we look for solutions close to the fixed point, say $\mathbf{x}^{(i)}$, write $\mathbf{x} = \mathbf{x}^{(i)} + \epsilon\mathbf{u}$, and find, to first order in ϵ, a linear system for $\mathbf{u}$:

$$\dot{\mathbf{u}} = D\mathbf{f}(\mathbf{x}^{(i)})\mathbf{u}, \tag{1.3}$$

where $D\mathbf{f}(\mathbf{x}^{(i)})$ is the Jacobian matrix of $\mathbf{f}$ evaluated at $x^{(i)}$,

$$D\mathbf{f}(\mathbf{x}^{(i)}) = \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix} = \begin{bmatrix} 2y^{(i)}\alpha & 1+2\alpha x^{(i)} \\ -1+2\gamma x^{(i)} & -2\beta y^{(i)} \end{bmatrix}. \tag{1.4}$$

The *linear eigenvalues* of the fixed point $\mathbf{x}^{(i)}$ are the eigenvalues λ_1, λ_2 of matrix $D\mathbf{f}(\mathbf{x}^{(i)})$. Together with the corresponding eigenvectors, they completely characterize the solutions of the linear system. Furthermore, if the real part of the eigenvalues does not vanish, then the fixed point is *hyperbolic* and Hartman-Grobman's theorem (Guckenheimer & Holmes, 1983) guarantees that the solutions of the nonlinear system (1.1) are locally *topologically conjugate* (see Definition 4.4) to the solutions of the linear system (1.3). That is, locally around the fixed point, there exists a homeomorphism mapping the solutions of one system to the solutions of the other which preserves their direction but not necessarily their parametrization. In other words, close to the fixed point, the solutions of the linear system provide an approximation to the solutions of the nonlinear system. In particular, if in the linear system the fixed point is hyperbolic and the origin is *asymptotically stable*[3] then the fixed point of the nonlinear system is also hyperbolic.

Let us apply these ideas to the special case $\alpha = \beta = 1$ where γ is considered as a control parameter. How do the solutions change as a function of γ? Since at the origin, $\mathbf{x}^{(1)} = (0,0)$, the linear eigenvalues are $\lambda_{1,2} = \pm i$, no information on the solutions of the nonlinear system can be obtained. However, the linear eigenvalues of the second fixed point, $\mathbf{x}^{(2)} = (1/\gamma, 0)$, located on the x-axis are $\lambda_{1,2} = \pm\sqrt{\gamma(2+\gamma)}/\gamma$. When $\gamma(2+\gamma) > 0$, the eigenvalues are real and nonvanishing. Hence, we conclude that for these values of γ, the fixed point is hyperbolic and the *stable and unstable manifolds* are of dimension 1. These manifolds are tangent to the *linear eigenvectors* (the eigenvectors of $D\mathbf{f}(\mathbf{x}^{(i)})$). We can perform a similar analysis around fixed points $\mathbf{x}^{(3,4)} = (-1/2, \pm\sqrt{2+\gamma}/2)$ to obtain their respective eigenvalues, $\lambda_{1,2} = \pm\sqrt{2+\gamma}$. Typically, if we were going to perform a dynamical systems analysis on this system, we would then use the local solutions around the hyperbolic fixed points to obtain the tangent approximation of the stable and unstable manifolds. Then, we would use all of the local information to obtain a global qualitative picture of the *phase portrait* constructed from the solution curves in the $x - y$ plane. A qualitative change of behavior of the solutions would be expected at $\gamma = 2$, identified as a *bifurcation point* of the system. The standard tools of bifurcation

[3]A fixed point $\bar{\mathbf{x}}$ is *asymptotically stable* if nearby solutions converge to $\bar{\mathbf{x}}$ as $t \to \infty$.

theory could then be used to describe the local nonlinear behavior of the solutions close to this point. In its entirety, this approach could be applied to a large class of systems where basic assumptions of hyperbolicity hold. However, rather than pursuing such a computation here, we examine this system from a different, *i.e.,* global, perspective by identifying global invariants.

1.1.2 An algebraic approach

There is no general approach to obtain an explicit form for the solutions of a system; furthermore, for most systems it can be shown that such forms do not exist. Therefore, we lower our expectations and attempt to identify global invariants. An invariant, $I(x, y, t)$, is a function which remains constant on all solution curves; that is, $I(x(t), y(t), t)$ is constant on all solutions $x = x(t), y = y(t)$ of the system. We refer to these invariants as *first integrals*. In the case of a two-dimensional system, having one first integral is enough to obtain a global picture of the solutions in the $x - y$ phase space. Moreover, if I is time-independent, then the solutions curves lie on the level set $I(x, y) = C$. Therefore, I completely characterizes the phase portrait. How do we find such useful quantities? Note that if a function $I(x, y)$ is constant for all $(x(t), y(t))$, then its derivative vanishes identically and we have (using the chain rule)

$$(\partial_x I)\dot{x} + (\partial_y I)\dot{y} = 0. \tag{1.5}$$

Using system (1.1), this equation becomes:

$$f_1\partial_x I + f_2\partial_y I = 0. \tag{1.6}$$

Therefore, we must choose a function I that satisfies this partial differential equation. A priori, candidate functions $I(\mathbf{x})$ can be chosen to be of any type, the simplest of which are polynomials in x and y. That is, for I in (1.6), we substitute in a polynomial of degree d with arbitrary coefficients for which we solve. The problem of finding a first integral then reduces to an algebraic problem, namely that of solving a system of linear equations for its parameters. For instance, let us try to find a quadratic first integral $I_2 = ax^2 + bxy + cy^2 + dx + ey$. Equation (1.6) then reads:

$$\begin{aligned}&b\gamma x^3 - 2\beta cy^3 + (2c\gamma + 4a\alpha)yx^2 + (2b\alpha - b\beta)\,y^2x + \\ &(e\gamma - b)x^2 + (b - e\beta)y^2 + (2a - 2c + 2d\alpha)yx - ex + dy = 0.\end{aligned} \tag{1.7}$$

Since a first integral must exist for all values of (x, y), every coefficient must vanish independently. This leads to an overdetermined system for the parameters (α, β, γ) and for the coefficients (a, b, c, d, e). This system has a unique non-trivial solution: $a = c, b = d = e = 0$ and $\gamma = -2\alpha, \beta = 0$ and we conclude that for these particular values of the parameters, $I_2 = x^2 + y^2$ is a first integral for the system. In this simple case, all solution curves lie on circles centered at the origin (see Figure 1.1).

A similar computation for the cubic case reveals that for $\beta = \alpha$, the system admits a cubic first integral $I_3 = 6\beta xy^2 - \gamma x^3 + 3y^2 + 3x^2$. We can now compare the results from the dynamical system analysis obtained in the special case $\alpha = \beta = 1$ with those results obtained from the analysis of I_3 in the case $\beta = 1$. The global phase portrait of the system (see Figure 1.2) shows how the fixed points $\mathbf{x}^{(3,4)}$ coalesce onto the fixed point $\mathbf{x}^{(2)}$. This second fixed point loses its hyperbolicity for values of γ between -2 and 0 where it becomes a center. Of particular interest are the surprising *heteroclinic solutions*. Heteroclinic solutions are orbits connecting one fixed point to another. Whenever they exist, analysis of the phase portrait reveals a pair of heteroclinic solutions connecting $\mathbf{x}^{(3)}$ with $\mathbf{x}^{(4)}$.

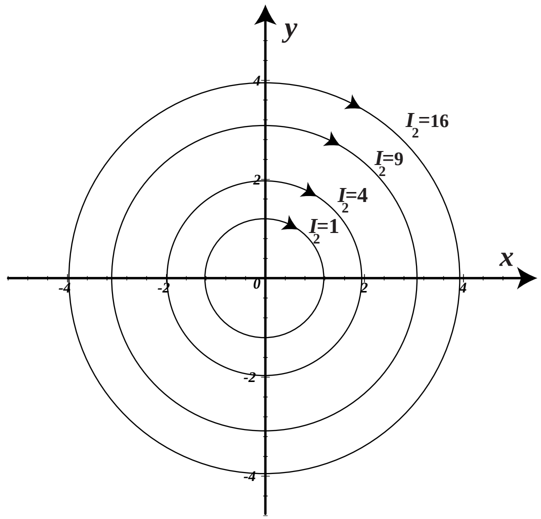

Figure 1.1: For $\beta = 0, \gamma = -2\alpha$ the solution curves lie on the level sets of $I_2 = x^2 + y^2$.

Moreover, one of these orbits is a straight line solution. In the particular case $\gamma = 1$, there is a cycle of straight heteroclinic solutions connecting $\mathbf{x}^{(3,4)}$ to $\mathbf{x}^{(2)}$. Clearly, these solutions play a special role since they delimit those orbits of qualitatively different behaviors; the three heteroclinic orbits form a triangle inside which orbits are bounded and periodic, and outside which they are unbounded. These heteroclinic connections are particular level sets of I_3, and we would like to know if they can be directly obtained without the first integral. We will come back shortly to this problem. We can continue this same computation to higher degrees and look for other first integrals, $I_4, I_5, I_6, \ldots$. However, if I_2 is a first integral of degree 2, then I_2^2 is a first integral of degree 4. Since these integrals are the same and do not provide additional information, we only look for new *independent first integrals.* We find that for

$$\begin{aligned}
\beta = 2\alpha, \quad & I_4 = -3\beta\gamma x^4 - 4(\gamma-\beta)x^3 + 6(\beta^2y^2+1)x^2 + 12y^2x\beta + 6y^2; \\
\beta = 3\alpha, \quad & I_5 = -24\beta^2\gamma x^5 - 30\beta(3\gamma-\beta)x^4 - 10(9\gamma - 4\beta^3y^2 - 12\beta)x^3 \\
& \quad + 45\left(4\beta^2y^2+3\right)x^2 + 270y^2x\beta + 135y^2; \\
\beta = 4\alpha, \quad & I_6 = -10\beta^3\gamma x^6 - 12\beta^2\left(6\gamma-\beta\right)x^5 - 15\beta\left(12\gamma - 6\beta - \beta^3y^2\right)x^4 \\
& \quad -40(4\gamma - 6\beta - 3\beta^3y^2)x^3 + 120(2+3\beta^2y^2)x^2 \\
& \quad +480y^2x\beta + 240y^2. \qquad (1.8)
\end{aligned}$$

This computation rapidly becomes intractable as the degree increases. However, we suspect that a first integral of degree $n+2$ might exist whenever $\beta = n\alpha$, that is, whenever the ratio β/α is an integer. Surprisingly, whenever $\beta = n\alpha$, the ratio λ_1/λ_2 of any pair of eigenvalues at one of the four fixed points is also an integer (either n or 1). Could there be a relation between the local analysis around the fixed points and the existence of a polynomial first integral? For instance, if the ratio of the eigenvalues is not an integer, could we conclude the non-existence of such first integrals? To prove or disprove our suspicion we need new tools to understand first integrals and how they appear. The key is in finding the appropriate building blocks of first integrals. Rather than looking at functions which are constant on all solutions, we look for functions which might be constant on their zero level

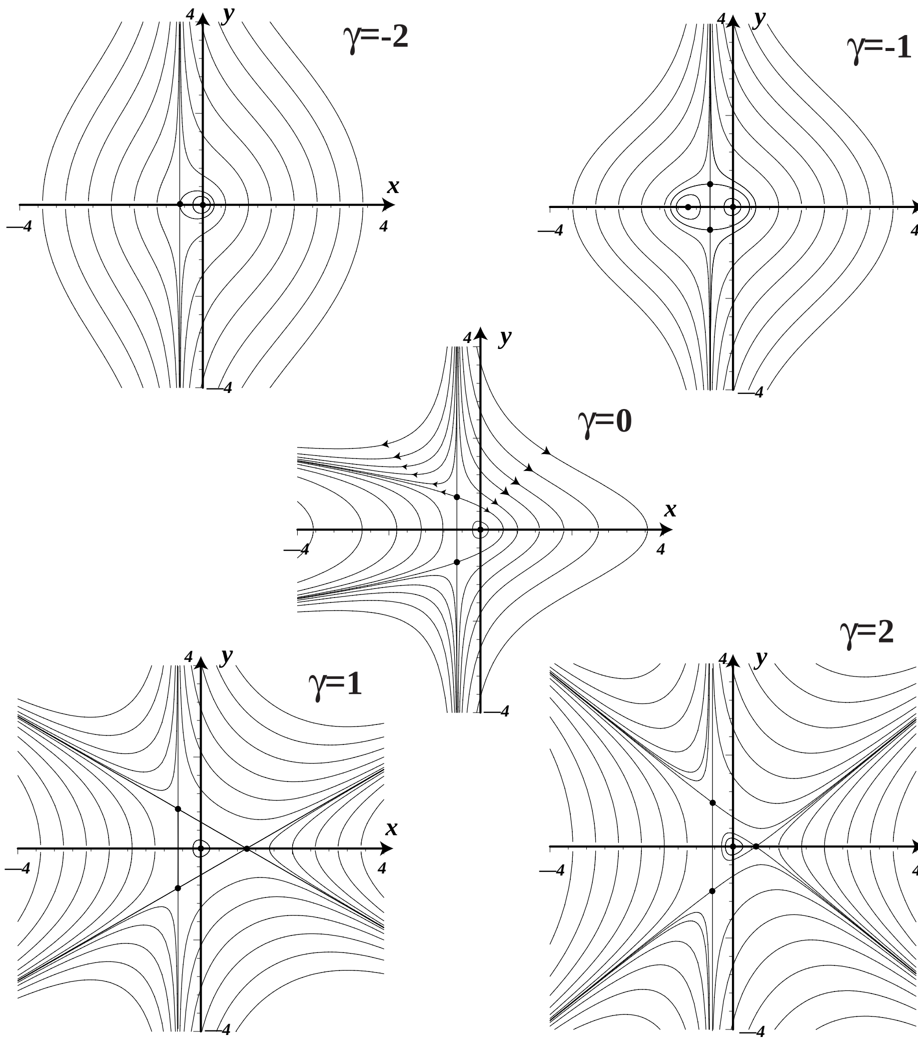

Figure 1.2: For $\beta = 1, \gamma = 0$ the solution curves lie on the level sets of I_3.

set. For instance, consider for Equation (1.1), the time derivative of the function $J_1 = 1 + 2\alpha x$:

$$\dot{J}_1 = f_1 \partial_x J_1 + f_2 \partial_y J_1 = 2y\alpha(1 + 2\alpha x) = 2y\alpha J_1. \tag{1.9.a}$$

We find that whenever $J_1(x, y) = 0$, we have $\dot{J}_1 = 0$, that is, the function $J_1(x, y)$ is invariant under the dynamics. Therefore, whenever $\alpha \neq 0$, the straight vertical line $x = -1/(2\alpha)$ is an invariant set. This invariant set can be seen in Figure 1.2. We call J_1 a *second integral* as it is invariant, but only on a restricted subset given by its zero level set. It is defined, in general, by $\dot{J}(\mathbf{x}) = \mu(\mathbf{x})J(\mathbf{x})$. In our example, the function μ is $\mu_1 = 2y\alpha$.

However, having a unique second integral is not enough to conclude the existence of a first integral; moreover, it fails to provide a global picture of the phase portrait. Therefore, we look for a second second integral. The procedure follows the method for finding first integrals; we look for a quadratic function $J_2 = J_2(\mathbf{x})$ such that $\dot{J}_2(\mathbf{x}) = \mu_2(\mathbf{x})J_2(\mathbf{x})$. Because the vector field is quadratic, the function $\mu_2(\mathbf{x})$ is at most linear since $\dot{J}_2$ is at most cubic. Again, this leads to a system of overdetermined equations whose solution is

$$J_2 = 2\beta(\alpha+\beta)(\beta+2\alpha)y^2 - 2\beta(\alpha+\beta)\gamma x^2 + 2\beta(\gamma+2\alpha+\beta)x - \gamma - 2\alpha - \beta \tag{1.10}$$

with $\mu_2 = -2\beta y$ The combined analysis of J_1 and J_2 reveals that *(i)* the fixed points $\mathbf{x}^{(3,4)}$ lie on

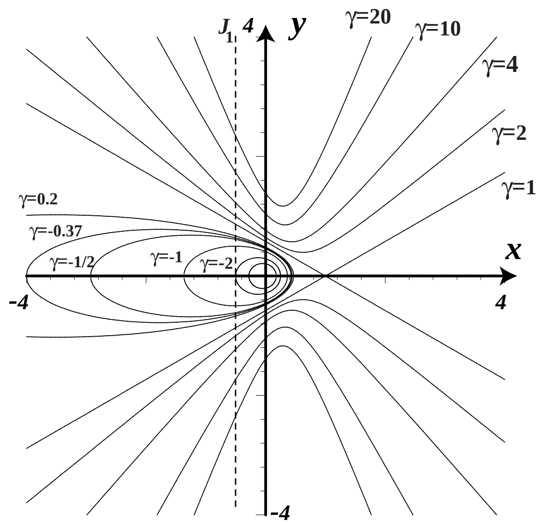

Figure 1.3: The level sets of the two second integrals, J_1 (dashed) and J_2 (solid), for $\alpha = 1$ and for varying values of γ. The fixed points $\mathbf{x}^{(3,4)}$ lie on the intersection of $J_1 = 0$ with $J_2 = 0$. This figure is not a phase portrait of the system since it represents curves for different values of the parameter γ.

the intersection of $J_1 = 0$ and $J_2 = 0$, and *(ii)* the heteroclinic orbits are on the level sets $J_1 = 0$ or $J_2 = 0$. These level sets are shown in Figure 1.3 for varying values of γ. For a fixed γ, the two level sets provide a clear picture of the global dynamics of the system; for instance, compare the level sets in Figure 1.3 for $\gamma = 1$ with the corresponding phase portrait of Figure 1.2. The second integrals in Figure 1.3 provide the essential skeleton for the phase portrait from which all other orbits can be qualitatively determined.

Actually, second integrals provide even more information. Consider the time derivative of the product:

$$I = J_1^{a_1} J_2^{a_2}. \tag{1.11}$$

Using the definition of $J_{1,2}$, we have

$$\begin{aligned} \dot{I} &= (a_1\frac{\dot{J}_1}{J_1} + a_2\frac{\dot{J}_2}{J_2})J_1^{a_1}J_2^{a_2}, \\ &= a_1\mu_1 + a_2\mu_2, \\ &= 2y(a_1\alpha - a_2\beta). \end{aligned} \tag{1.12}$$

Therefore, if we choose $a_1 = \beta$ and $a_2 = \alpha$, $\dot{I} = 0$ and we conclude that $I = J_1^{\beta} J_2^{\alpha}$ is a first integral for all values of the parameters. We can now answer the question about the existence of polynomial first integrals. If $q\beta = p\alpha$ with q, p positive integers, $I^{\frac{q}{\alpha}} = J_1^p J_2^q$ is a polynomial integral of degree

$p+2q$. Moreover, if p, q are integers such that $pq < 0$, then $I^{q/\alpha}$ is a *rational first integral*. In general, if β/α is not rational, then I is a *transcendental first integral*. For this problem, the second integrals not only provide a skeleton for the phase portrait, but also serve as the building blocks of the first integral and allow for a complete resolution of the problem for all parameter values. However, the first integral does not completely describe the dynamics for it only provides information about the solution curves in phase space; for instance, the periods of the periodic orbits cannot be obtained from the first integral and further analysis is required for their determination.

This example was carefully chosen for the purpose of illustrating the concepts of first and second integrals. In general, we do not expect to have a first or even a single second integral for a generic system of differential equations. Nevertheless, this type of analysis is so fruitful that a general approach to prove or disprove the existence of such functions is needed. In general, we wish to identify those values of the parameters for which first or second integrals exist. A perturbative analysis of the phase portrait around these particular cases could then be performed.

1.1.3 An analytic approach

There is yet a different way to analyze the solutions of Equations (1.1), that is, to look at their local behavior around their singularities in *complex time*. This type of analysis might seem strange at first. Why would complex time behavior carry any information on real-time behavior? This theory is actually an extension of *Frobenius' method*, one of the basic tools for studying linear differential equations with time-dependent coefficients. The following analysis is a nonlinear version of this method. Assume that a solution $\mathbf{x} = \mathbf{x}(t)$ of system (1.1) is known. This solution can be analytically continued for complex values of the time t and may exhibit some singularities in complex time. A natural question would be to determine the type of each singularity. If we were given an explicit form of the solution, a local expansion around each singularity would reveal whether it is a pole, a branch point, or an essential singularity. However, if such a form is not available, this information can still be obtained. Assume that as $x(t)$ and $y(t)$ get close to the singularity, they behave like $x(t) \sim a_0 t^p$ and $y(t) \sim b_0 t^q$ where p or q is negative in order to represent singular behavior.[4] If we substitute this ansatz into (1.1), we obtain

$$a_0 p t^{p-1} = b_0 t^q + 2\alpha a_0 b_0 t^{p+q}, \tag{1.13.a}$$
$$b_0 q t^{q-1} = -a_0 t^p + \gamma {a_0}^2 t^{2p} - \beta {b_0}^2 t^{2q}. \tag{1.13.b}$$

Clearly, $x(t) = a_0 t^p$, $y(t) = b_0 t^q$ is not an exact solution. However, as $t \to 0$, we can neglect the terms t^p and t^q since they are dominated by the other terms. We thus obtain

$$a_0 p t^{p-1} = 2\alpha a_0 b_0 t^{p+q}, \tag{1.14.a}$$
$$b_0 q t^{q-1} = \gamma {a_0}^2 t^{2p} - \beta {b_0}^2 t^{2q}. \tag{1.14.b}$$

This reduced system has a non-trivial solution if we choose $p = q = -1$ and $a_0 = \pm\frac{\sqrt{\beta+2\alpha)/\gamma}}{(2\alpha)}$, $b_0 = -1/(2\alpha)$. Note that there are two possible choices for a_0, corresponding to the fact that the solution $\mathbf{x}(t)$ may behave differently around different singularities (much like the function $\operatorname{sech}(t)$ behaves differently around the two singularities $t_* = \pm i\pi/2$). Also, because the system does not depend explicitly on time, any solution can be shifted by an arbitrary time. Therefore, we write $x(t) \sim$

[4] The symbol $\sim$ is to be understood as the dominant behavior of the function as $t \to 0$. These notions will be carefully defined in Chapter 3.

$a_0(t-t_*)^{-1}, y(t) \sim b_0(t-t_*)^{-1}$ and conclude that the dominant behavior of the solutions around the *movable singularity* t_* is $(t-t_*)^{-1}$. However, this is not enough to conclude that the singularities of the solutions are poles since higher order terms which were neglected in this simple computation can still introduce branch points.

To understand the behavior of the solutions, we build local series solutions around their singularities. *Laurent series* are the simplest such series which are compatible with the solutions' dominant behavior. The above computation shows that the first term of these series are $x(t) \sim a_0(t-t_*)^{-1}, y(t) \sim b_0(t-t_*)^{-1}$ and we seek a local solution of the form:

$$x(t) = (t-t_*)^{-1}\left(a_0 + \sum_i^{\infty} a_i (t-t_*)^i\right), \tag{1.15.a}$$

$$y(t) = (t-t_*)^{-1}\left(b_0 + \sum_i^{\infty} b_i (t-t_*)^i\right), \tag{1.15.b}$$

where coefficients (a_i, b_i) are to be determined. Consider again the particular case $\beta = \alpha = \gamma = 1$ for which a polynomial first integral of degree 3 exists. We look for a solution around the singularity determined by the behavior $a_0 = \sqrt{3}/2, b_0 = -1/2$. To do so, we substitute expansion (1.15) into system (1.1), and by identifying terms of the same power of $(t-t_*)$, we compute coefficients (a_i, b_i). For instance, if we collect the factors of $(t-t_*)^{-1}$, we find a linear system for coefficients (a_1, b_1):

$$-a_1 + \sqrt{3}b_1 = \frac{1}{2}, \quad b_1 + \sqrt{3}a_1 = \frac{\sqrt{3}}{2}, \tag{1.16}$$

whose solution is $a_1 = 1/4, b_1 = \sqrt{3}/4$. At each order in $(t-t_*)$, we find a closed system for coefficients (a_i, b_i). The Laurent solution up to order 4 reads:

$$\begin{aligned} x = (t-t_*)^{-1}\Bigg[&\frac{\sqrt{3}}{2} + \frac{1}{4}(t-t_*) + \frac{\sqrt{3}}{8}(t-t_*)^2 + \frac{\sqrt{3}}{3}b_3(t-t_*)^3 \\ &+ \left(\frac{2}{5}b_3 - \frac{\sqrt{3}}{160}\right)(t-t_*)^4 + O\big((t-t_*)^5\big)\Bigg], \end{aligned} \tag{1.17.a}$$

$$\begin{aligned} y = (t-t_*)^{-1}\Bigg[&-\frac{1}{2} + \frac{\sqrt{3}}{4}(t-t_*) - \frac{1}{8}(t-t_*)^2 + b_3(t-t_*)^3 \\ &+ \left(\frac{1}{160} - \frac{2\sqrt{3}}{15}b_3\right)(t-t_*)^4 + O\big((t-t_*)^5\big)\Bigg]. \end{aligned} \tag{1.17.b}$$

This series can be computed to arbitrary order and an analysis of the linear system for (a_i, b_i) reveals that it is indeed a Laurent series. Note that to the coefficient b_3 is arbitrary. Hence, the series contains two arbitrary constants (t_* and b_3) and is therefore a local expansion of the general solution. Note also that the arbitrary constant appears at the same order (order 3) as the degree of the first integral.

To obtain a better understanding of the structure of these series, we find a local expansion in the form of (1.15) for the general case. If we substitute this solution into system (1.1), we obtain a linear recursion relation for the coefficients $\mathbf{a}_i = (a_i, b_i)$ of the form

$$K\mathbf{a}_i = i\mathbf{a}_i + \mathbf{P}_i(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{i-1}), \tag{1.18}$$

where $\mathbf{P}_i$ is a vector polynomial in its arguments and K is the so-called *Kovalevskaya matrix*:

$$K = \begin{bmatrix} 1+2\alpha b_0 & 2\alpha a_0 \\ 2\gamma a_0 & 1-2\beta b_0 \end{bmatrix}. \tag{1.19}$$

The eigenvalues of K, called the *Kovalevskaya exponents*, play a central role in the analysis of solutions around their singularities, similar to that of linear eigenvalues in the analysis of solutions around their fixed points. In our example, the Kovalevskaya exponents are $\rho_1 = -1$ and $\rho_2 = 2 + \beta/\alpha$. Note that, whenever $\beta = n\alpha$, there is a first integral of degree $2 + \beta/\alpha$. Is this an accident or is there a general relationship between the degrees of the first integrals and the Kovalevskaya exponents?

Assume that $\beta = n\alpha$ with n integer, then the linear recursion relation can be solved up to $i = \rho_2 = n + 1$ (since $(K - i\mathrm{I})$ is invertible, we can solve for $\mathbf{a}_i = (K - i\mathrm{I})^{-1}\mathbf{P}_i$). Now, for $i = \rho_2$,

$$(K - \rho_2\mathrm{I})\mathbf{a}_{\rho_2} = \mathbf{P}_{\rho_2}(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{n+1}), \tag{1.20}$$

and the Fredholm alternative implies that this linear system has a solution if and only if eigenvector $\bar{\boldsymbol{\beta}}_2$ of K^T of eigenvalue ρ_2 is such that

$$\bar{\boldsymbol{\beta}}_2 . \mathbf{P}_{\rho_2}(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{n+1}) = 0. \tag{1.21}$$

If this *compatibility condition* is not satisfied, the linear system in (1.20) does not have a solution and there is no local Laurent series solution for our system. Therefore, a more general series with logarithmic terms must be introduced. However, in our particular case, an explicit computation shows that the compatibility condition is always satisfied and that $\mathbf{a}_{\rho_2}$ is defined up to an arbitrary multiplicative constant. For all $i > \rho_2$, the linear recursion relation can be trivially solved and therefore we conclude that if $\alpha = n\beta$, the system admits a 2-parameter, hence general, Laurent series solution locally around the singularity t_*. The general solution is therefore locally single-valued and the singularities are simple poles. The two parameters are the arbitrary location of t_* and the arbitrary constant appearing in the computation of $\mathbf{a}_{\rho_2}$. Surprisingly, whenever the system admits a polynomial first integral, it also admits a two-parameter family of Laurent solutions.

If β/α is not an integer, then we can always solve the linear recursion relation in (1.18) and find a one-parameter family solution. What type of information can be obtained from these series, and how are they related to the existence of second integrals? Could these particular solutions be the local expansions of the solution lying on the level set $J_2 = 0$?

1.1.4 Relevant questions

Based on the observations obtained during the analysis of this simple two-dimensional example, we can articulate some general questions on the integrability of dynamical systems.

1. **First integrals.** Why did we choose polynomial first integrals? What are the possible types of first integrals for general vector fields? Do polynomial first integrals play a particular role in the analysis? Is there a general algorithm to obtain them? (Chapter 2) Can we develop a formalism for vector fields that is more suitable for such a construction? (Chapter 4).
2. **Second integrals.** These invariants have appear to be important quantities for they allow us to build first integrals and their zero level sets are heteroclinic or homoclinic orbits. Is this true in general? Can we always build a first integral if sufficiently many second integrals are known and if so, how many second integrals are needed? (Chapter 2).

3. **Local analysis around fixed points.** Local analysis around fixed points provides us with linear eigenvalues. We saw that the ratio of these eigenvalues corresponds to the degree of the first integrals. Is this a general relationship? Can we use such an analysis to prove the existence of a first integral? This is unlikely since the linear eigenvalues only reflect the local behavior of the solution and nothing, a priori, can be said about its global behavior. However, the existence of a global invariant has local implications and we might be able to use local analysis to prove the *non-existence* of first integrals. (Chapter 5).
4. **Singularity analysis.** Is there a general way to analyze the solutions of a system of differential equations without knowing their explicit solutions? What are the possible types of singularities? In some cases, the solutions are particularly simple (Laurent series). Can we use this fact to define a notion of integrability? (Chapter 3) How is this notion related to the existence of first integrals? What is the general relationship between Kovalevskaya exponents and the degree of first integrals? (Chapter 5).

1.2 The Lorenz system

There is probably no single system that has been more analyzed than the Lorenz system (Lorenz, 1963). In its traditional form, it reads:

$$\dot{x} = \sigma(y - x), \tag{1.22.a}$$
$$\dot{y} = rx - y - xz, \tag{1.22.b}$$
$$\dot{z} = xy - \beta z, \tag{1.22.c}$$

where σ, r and β are arbitrary parameters (usually assumed positive). In the following, we will also use the alternative form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} = (x, y, z)$ and $\mathbf{f} = (\sigma(y - x), rx - y - xz, xy - \beta z)$. First introduced by Lorenz as a Galerkin reduction of Boussinesq equations for the modeling of Rayleigh-Benard convection, this system has become a paradigm for the study of chaos in low-dimensional autonomous dissipative systems. An excellent account of its history, its derivation in different contexts, and its physical interpretation can be found in Strogatz (1994).

1.2.1 A dynamical system approach

The Lorenz system is extremely rich and literally hundred of papers have been devoted to its analysis (see for instance Sparrow (1982) or Jackson (1992)). In this section, only those key features that are relevant for the rest of our discussion are mentioned. The first important feature of the Lorenz system is its dissipative nature. If we compute the divergence of the vector field:

$$\begin{aligned} \partial_{\mathbf{x}}.\mathbf{f} &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}, \\ &= -(\sigma + \beta + 1), \end{aligned} \tag{1.23}$$

we can readily conclude that whenever $\sigma + \beta + 1 > 0$, any initial volume $V(t = 0)$ in the xyz-phase space contracts in time, that is, $\dot{V} < 0$.[5] The second key feature of the Lorenz system is the symmetry $x \to -x, y \to -y, z \to z$

[5] This result is a direct consequence of Liouville's theorem on the contraction of phase space volume given in Exercise 3.3.

For historical reasons, the analysis of the Lorenz system is usually performed for the parameter values $\sigma = 10, \beta = 8/3$ and r varying from 0 to ∞ for which there are at most three fixed points (one if $r < 1$, two if $r = 1$ and three if $r > 1$) located at $\mathbf{x}^0 = (0,0,0)$ and $\mathbf{x}^{(1,2)} = (\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, r-1)$. The analysis of the linear eigenvalues of the fixed points together with extensive numerical investigations have revealed the following behavior for increasing values of r (see for instance Jackson (1992)):

1. For $r < 1$, the origin is globally stable.

2. For $1 < r < r_1$ the origin becomes unstable and two new fixed points $\mathbf{x}^{(1,2)}$ are stable with real eigenvalues. The value of $r_1 \approx 1.34$ is given by the positive root of $96r^3 + 54119r^2 + 91470r - 22130 = 0$.

3. For $r_1 < r < r_4$, the two fixed points $\mathbf{x}^{(1,2)}$ are stable with eigenvalues of non-vanishing imaginary parts (causing any initial point to spiral into a fixed point under the dynamics of the system). For $r_4 = 470/9 \approx 24.74$, the system undergoes a subcritical Hopf bifurcation and there exists a couple of unstable limit cycles for $r_2 < r < r_3$ where $r_2 \approx 13.926$.

4. For $r = r_2 \approx 13.926$ the unstable manifold of the origin $W^{\mathrm{u}}(\mathbf{0})$ intersects with the stable manifold of $\mathbf{x}^{(1,2)}$ and there are two heteroclinic connections (these connections are very hard to exactly pinpoint since they are unstable, but as r increases to 13.96 the unstable manifold of $\mathbf{0}$ gets closer and closer to the fixed points in a spiraling motion). This global bifurcation is called a *homoclinic explosion*

5. For $r_2 < r < r_3 \approx 24.06$, the fixed points $\mathbf{x}^{(1,2)}$ are stable; however, the time it takes for any initial conditions to reach a fixed point increases as r increases to 24.06. The transient trajectory is very erratic, hence the name *transient chaos*. As $r \approx 24.06$, it takes an infinite time to reach a fixed point and it is believed that a strange attractor appears at this value.

6. For $24.06 < r < 24.74$, both the strange attractor and the fixed points $\mathbf{x}^{(1,2)}$ are stable. As $r \to r_4$, the basins of attraction of the fixed points shrink to zero and the fixed points become unstable.

For these particular parameter values, this complicated scenario illustrates the complexity of behavior in dynamical systems. Rather than following this analysis, we look for global invariants. It is not clear a priori that a global analysis of the Lorenz system can help us understand some of these behaviors or give us any information at all.

1.2.2 An algebraic approach

Independently of its rich dynamical behaviors, the Lorenz system surprisingly has very interesting integrability features which were first studied by Segur (1982) by singularity analysis (see next section), and then by Kús (1983) by the Carleman embedding method. Since then, it has become a testing ground for all new (and old) methods in integrability theory (Schwarz, 1985; Strelcyn & Wojciechowski, 1988; Schwarz, 1991; Giacomini *et al.*, 1991; Goriely, 1996).

Again, the first step in determining the existence of a global invariant is to look for polynomial first integrals. For instance, we can look for a quadratic first integral $I_2 = I_2(x, y, z)$ following the simple method shown in the previous section. However, we would soon realize that the existence of such a first integral is incompatible with the dissipative nature of the Lorenz system.[6] The way to take this characteristic into account is to consider *time-dependent first integrals* of the form $I_d(x,y,z,t) = P_d(x,y,z)e^{\gamma t}$, where P_d is a polynomial of degree d and γ is a constant to be determined. For instance, we look for a quadratic time-dependent first integral. Again, using the condition $\dot{I}_2 = e^{\gamma t}\left(\mathbf{f}.\partial_{\mathbf{x}} P_2 + \gamma P_2\right) = 0$, we obtain an overdetermined system of equations for the coefficients of P_2 and γ. The solvability condition for the existence of a solution imposes some conditions on the parameters of the system (β, σ and r). For instance, if $\beta = 2\sigma$, then $I_2^{(1)} = (x^2 - 2\sigma z)e^{2\sigma t}$ is a first integral. This analysis can be performed degree by degree. However, this procedure rapidly becomes extremely cumbersome and clearly other methods have to be found to determine the existence of first integrals. (All of the known first integrals of the Lorenz system are given in Table 1.1.)

(β, σ, r)	First integral
$(2\sigma, \sigma, r)$	$I_2^{(1)} = (x^2 - 2\sigma z)e^{2\sigma t}$
$(1, \sigma, 0)$	$I_2^{(2)} = (x^2 + y^2)e^{2t}$
$(1, 1, r)$	$I_2^{(3)} = (-rx^2 + y^2 + z^2)e^{2t}$
$(6\sigma - 2, \sigma, 2\sigma - 1)$	$I_4^{(1)} = \left(\frac{(2\sigma-1)^2}{\sigma}x^2 + \sigma y^2 - (4\sigma - 2)xy - \frac{1}{4\sigma} + x^2 z\right) e^{4\sigma t}$
$(0, \frac{1}{3}, r)$	$I_4^{(2)} = \left(-rx^2 + \frac{1}{3}y^2 + \frac{2}{3}xy + x^2 z - \frac{3}{4}x^4\right) e^{\frac{4}{3}t}$
$(4, 1, r)$	$I_4^{(3)} = \left(4(r-1)z - rx^2 - y^2 + 2xy - x^2 z + \frac{1}{4}x^4\right) e^{4t}$

Table 1.1: The known first integrals of the Lorenz system

Note that we can combine the two cases $\beta = 2\sigma$ together with $\beta = 1,\ r = 0$ to obtain the existence of two first integrals when $\beta = 1,\ \sigma = 1/2,\ r = 0$. These two time-dependent first integrals can be combined to create a time-independent rational first integral $I = (x^2 - z)^2/(x^2 + y^2)$.

The first integrals of the Lorenz system exist only for very restricted values of the parameters. Therefore, they may not be of help in the study of the dynamical behavior of the system, say in the case $\sigma = 10, \beta = 8/3$. Nevertheless, the structure of the polynomials give us a clue on the general structure of the phase space and indeed can be used as Lyapunov functions. For instance, for σ and β arbitrary, we can consider the evolution in phase space of initial conditions on an ellipsoid obtained by considering the general coefficient of the polynomial appearing in $I_2^{(3)}$, that is,

$$V = ax^2 + by^2 + cz^2. \tag{1.24}$$

If a, b and c are positive parameters, the level sets $V(x,y,z) = C$ are concentric ellipsoids centered at the origin. We want to determine a, b, c and the values of r such that the trajectories based on any given ellipsoid always enter the ellipsoid. If such values can be found for all ellipsoids, all trajectories eventually reach the (globally stable) fixed point at the origin (see Figure 1.4). To do so, we determine

[6] In general, dissipative systems can support first integrals. In the particular case of the Lorenz system, one can readily show, using Theorem 5.1, that the existence of a polynomial first integral is incompatible with the linear eigenvalues at the origin.

the parameter values so that that $\dot{V} < 0$. For instance, by choosing $a = 1/\sigma,\ b = 1,\ c = 1$, we find:

$$\begin{aligned} \dot{V} &= 2\left(\frac{1}{\sigma}x\dot{x} + y\dot{y} + z\dot{z}\right), \\ &= -2\left(x - \frac{r+1}{2}y\right)^2 + -2\left[1 - \left(\frac{r+1}{2}\right)^2\right]y^2 - 2\beta z^2. \end{aligned} \tag{1.25}$$

Clearly, if $r < 1$, $\dot{V} < 0$ and we can conclude that the origin is globally stable.

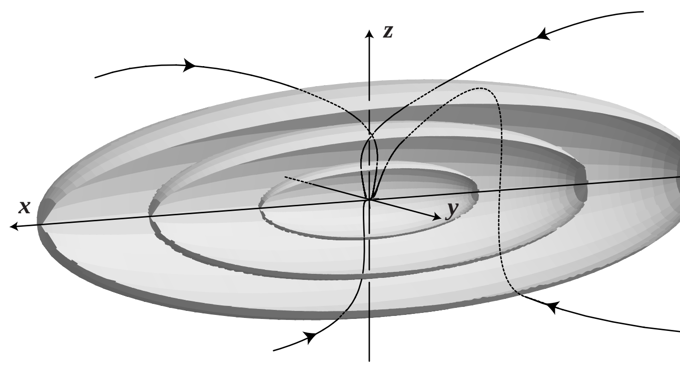

Figure 1.4: The level sets of the Lyapunov function $V(x, y, z) = C$ for $\beta = 8/3, \sigma = 10, r = 1/2$ and $C = 1, 4, 9$, shown together with some orbits of the Lorenz equations.

The first integrals of the Lorenz system can be used to find the general form of the Lyapunov functions which, in turn, can be used to prove the global stability of a fixed point. However, we are far from obtaining some information on the chaotic nature of the dynamics as sketched in the previous sections. Can we modify the notion of Lyapunov functions to obtain a global bound on the invariant set of the Lorenz system? Can we use first integrals to obtain such bounds (Doering & Gibbon, 1995; Giacomini & Neukirch, 1997; Neukirch & Giacomini, 2000)?

1.2.3 An analytic approach

We now analyze the solutions of the Lorenz equations in complex time (Segur, 1982; Tabor & Weiss, 1981; Levine & Tabor, 1988). To do so, we look for those solutions around a singularity where the solution blows up. To determine the dominant behavior of the solutions close to these singularities, we substitute

$$x(t) = a_0(t - t_*)^p, \quad y(t) = b_0(t - t_*)^q, \quad z(t) = c_0(t - t_*)^r, \tag{1.26}$$

into the Lorenz system (1.22) and balance the dominant terms as $t \to t_*$. In doing so, we obtain $\mathbf{p} = (p, q, r) = (-1, -2, -2)$ and $\mathbf{a}_0 = (a_0, b_0, c_0) = (\pm 2i, \mp 2i/\sigma, -2/\sigma)$. We have two possible choices for $\mathbf{a}_0$ because a solution in complex time has two possible different expansions around different singularities. We now look for a local solution around the singularity of the form

$$x(t) = (t-t_*)^{-1}\left(a_0 + \sum_{j=1}^{\infty} a_j (t-t_*)^j\right), \tag{1.27.a}$$

$$y(t) = (t-t_*)^{-2}\left(b_0 + \sum_{j=1}^{\infty} b_j (t-t_*)^j\right), \tag{1.27.b}$$

$$z(t) = (t-t_*)^{-2}\left(c_0 + \sum_{j=1}^{\infty} c_j (t-t_*)^j\right). \tag{1.27.c}$$

If we substitute this solution into the equations, we obtain a set of recursion relations for the coefficients $\mathbf{a}_j = (a_j, b_j, c_j)$ of the form

$$K\mathbf{a}_j = j\mathbf{a}_j + \mathbf{P}_j(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{j-1}), \tag{1.28}$$

where

$$K = \begin{bmatrix} 1 & \sigma & 0 \\ \frac{2}{\sigma} & 2 & -2i \\ \frac{-2i}{\sigma} & 2i & 2 \end{bmatrix}, \quad \mathbf{P}_j = \begin{bmatrix} \sigma a_{j-1} \\ \sum_{k=1}^{j-1} a_k c_{j-k} + r a_{j-1} - b_{j-1} \\ \sum_{k=1}^{j-1} a_k b_{j-k} - \beta c_{j-1} \end{bmatrix}. \tag{1.29}$$

The matrix K has three integer eigenvalues: $-1, 2, 4$. Therefore, when $j = 2$ or $j = 4$, the matrix $(K - j\mathrm{I})$ is singular and the linear system for $\mathbf{a}_2$ and $\mathbf{a}_4$ does not have a solution in general. The condition for the existence of a solution for $\mathbf{a}_2$ and $\mathbf{a}_4$ is given by the Fredholm alternative: If $\bar{\boldsymbol{\beta}}^{(2)}$ and $\bar{\boldsymbol{\beta}}^{(4)}$ are the eigenvectors of K^T, then the linear system (1.28) has a solution only if

$$\bar{\boldsymbol{\beta}}^{(2)}.\mathbf{P}_2 = 0 \quad \text{and} \quad \bar{\boldsymbol{\beta}}^{(4)}.\mathbf{P}_4 = 0. \tag{1.30}$$

These conditions are satisfied only for particular values of the parameters β, σ, and r, namely:

$$(\beta - 2\sigma)(\beta + 3\sigma - 1) = 0, \tag{1.31.a}$$

$$\sigma^2(21 - 3\beta) + \beta^2(13 + 2\sigma) + 44\sigma + 4\beta - 55\beta\sigma - 17 = 0, \tag{1.31.b}$$

$$\begin{aligned} &\frac{4}{3}(1 + 2\beta - 3\sigma)^2 \left[(1 - \beta + 3\sigma)^2 + (1 - \beta + 3\sigma)(5 + 9\sigma + 7\beta) \right.\\ &\qquad\qquad\qquad \left. +6\sigma(4 + 2\beta - 6\sigma)\right] \\ &\quad + 16\beta(1 - \beta + 3\sigma)(1 + 2\beta - 3\sigma)(1 + \beta - 3\sigma) \\ &\quad + (r - 1)\left[72\beta\sigma(5 + \beta - 3\sigma\right] = 0. \end{aligned} \tag{1.31.c}$$

The first condition in (1.31) is obtained at $j = 2$. The others, obtained at $j = 4$, exist since $\mathbf{P}_4$ depends on one arbitrary constant (introduced at $j = 2$) and the condition $\bar{\boldsymbol{\beta}}^{(4)}.\mathbf{P}_4 = 0$ should be satisfied for all values of this arbitrary constant. Unless these three equations for β, σ, r are identically satisfied, there is no local expansion of the general solution of the form (1.27), that is, the general solution cannot be expanded in Laurent series. These conditions are satisfied for three sets of parameter values:

1. $(\beta, \sigma, r) = (1, 1/2, 0)$. This is the case found in the previous sections where the system admits two time-dependent first integrals ($I_2^{(1)}$ and $I_2^{(2)}$). Furthermore, it can be shown that the solutions can be expressed in terms of *Jacobi elliptic functions*.

2. $(\beta, \sigma, r) = (2, 1, 1/9)$. The system admits one first integral $I_2^{(1)}$ that can be used to reduce the system to the *second Painlevé equation* (see Segur (1982)).

3. $(\beta, \sigma, r) = (0, 1/3, r)$. The system admits one first integral $I_4^{(2)}$ that can be used to reduce the system to the *third Painlevé equation* (see Segur (1982)).

These three particular cases are those where the Lorenz system is *completely integrable* in the sense that the solutions can be expressed in terms of known functions (elliptic functions or Painlevé transcendents). At first sight, it is rather surprising that those specific values where the local solutions around complex time singularities are Laurent series, hence single-valued, also correspond to the cases where first integrals exist and the system is integrable. Furthermore, in the cases where other first integrals exist, combinations of conditions (1.31) are also satisfied. This interesting connection is the basis of the singularity analysis for integrable systems and will be further investigated in this book.

What happens if conditions (1.31) are not satisfied? Clearly, the solutions are not single-valued since they cannot be expanded in Laurent series and the ansatz (1.27) is not valid. What then is the correct expansion of the solution close to a singularity? Here, since the eigenvalues of matrix K are integer, the singularity is a *logarithmic branch point* and that the correct expansion must involve logarithmic corrections of the form (Levine & Tabor, 1988):

$$x(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}(t-t_*)^{j-1}\left[(t-t_*)^2\log(t-t_*)\right]^k, \tag{1.32.a}$$

$$y(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{jk}(t-t_*)^{j-2}\left[(t-t_*)^2\log(t-t_*)\right]^k, \tag{1.32.b}$$

$$z(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}(t-t_*)^{j-2}\left[(t-t_*)^2\log(t-t_*)\right]^k. \tag{1.32.c}$$

Again, linear recursion relations for the coefficients (a_{jk}, b_{jk}, c_{jk}) can be obtained and solve order by order with no extra conditions on the parameters. These series, known as Ψ*-series*, then can be analyzed

1.2.4 Relevant questions

1. **First integrals.** It seems that the Lorenz system only supports polynomial first integrals (and one rational first integral which is really a combination of two polynomial integrals). Why do polynomial integrals play such an important role for dynamical systems? How can we compute them effectively or prove their non-existence? How can we rule out the existence of other types of first integrals? How can we use these first integrals to obtain some information on the global dynamics for values of the parameters where they do not exist? If we know the first integrals, is there an efficient way to build scalar functions, such as Lyapunov functions, which control the trajectories? See Chapter 2.

2. **Singularity analysis.** The analysis of local solutions around singularities reveals that whenever solutions are locally single-valued, the system is completely integrable. Is this a general property? It appears that these local solutions can be computed algorithmically (at least for the Lorenz system). How can we find, in general, all of the possible local solutions? Did we miss important solutions with our ansatz? How can we show that the general solution is globally single-valued? How is this property related to the integrability of the system? What can we say about the dynamics of a system when the local solutions cease to be single-valued? What is the dynamical information contained in these Ψ-series? These questions will be answered in Chapters 3,4,5,7.

1.3 Exercises

1.1 Consider system (1.1) with $\beta = 2\alpha$ and discuss the different bifurcations and changes in the phase portrait by analyzing the first integral I_4.

1.2 Consider the solution of system (1.1) on the vertical line $x = -\frac{1}{2\alpha}$. Show that it reduces to the equation, $\dot{y} = \frac{(2\alpha+\gamma)}{4\alpha}$, whose solution is

$$y(t) = \frac{\sqrt{2\beta\alpha + \beta\gamma}}{2\alpha\beta} \tanh\left(\frac{\sqrt{2\beta\alpha + \beta\gamma}\,(t + C)}{2\alpha}\right). \tag{1.33}$$

Find the singularities of this solution. Find a local expansion around these singularities and compare this solution with the general form in (1.15). Can you explain why these solutions are different? Did we miss a local solution?

1.3 Consider system (1.1) for the particular case $\alpha = \beta = \gamma = 1$. Show that the second integral is $J_2 = (y - \frac{\sqrt{3}}{3}(x-1))(y + \frac{\sqrt{3}}{3}(x-1))$ and the level set $J_2 = 0$ are straight lines. On each of these straight lines, find an equation for $x(t)$ and solve it. Show that the general solution for $x(t)$ has a local expansion of form (1.15).

1.4 Show that if a three-dimensional system, such as the Lorenz system, has negative divergence ($\partial_{\mathbf{x}}.\mathbf{f} < 0$) everywhere, there cannot exist a pair of time-independent first integrals. Could there be a single first integral? Find a simple three-dimensional system which admits a polynomial first integral but has negative divergence.

1.5 Show that the existence of a time dependent first integral, $I_d(x, y, z, t) = P_d(x, y, z)e^{\gamma t}$, implies the existence of a second integral (which one?).

1.6 For the Lorenz system, compute the linear eigenvalues $\boldsymbol{\lambda}^{(i)} = (\lambda_1, \lambda_2, \lambda_3)$ at the origin and show that whenever there exists a first integral of the form $I_d(x, y, z, t) = P_d(x, y, z)e^{\gamma t}$, the linear eigenvalues are such that there exists a vector of positive integer numbers $\mathbf{m}$ such that $\mathbf{m}.\boldsymbol{\lambda} + \gamma = 0$. Is this true for the linear eigenvalues around the other fixed points?

1.7 There is yet another integrable case of the Lorenz equations obtained in the limit $r \to \infty$ (Sparrow, 1982). This case can be obtained by introducing the following change of variables:

$$r = \frac{1}{\sigma\epsilon^2}, \quad x = \frac{x'}{\epsilon}, \quad y = \frac{y'}{\sigma\epsilon^2}, \quad z = \frac{z'}{\sigma\epsilon^2}, \quad t = \epsilon t', \tag{1.34}$$

which maps the Lorenz system (1.22) (after dropping the primes) to:

$$\dot{x} = y - \sigma\epsilon x, \tag{1.35.a}$$
$$\dot{y} = x - xz - \epsilon y, \tag{1.35.b}$$
$$\dot{z} = xy - \epsilon\beta z. \tag{1.35.c}$$

(i) Take the limit $\epsilon = 0$; *(ii)* show that the new system is conservative, that is, the divergence vanishes identically; *(iii)* show that the system (for $\epsilon = 0$) has two time-independent quadratic first integrals, $I_1 = C_1$ and $I_2 = C_2$; *(iv)* use these two first integrals to obtain a first order equation for x. *(v)* Show that the solution of this equation can be written in terms of elliptic functions (Robbins, 1979). *(vi)* Find the values of the arbitrary constants in the first integrals such that the intersection of the two level sets $I_1 = C_1$ and $I_2 = C_2$ is either one periodic orbit, two periodic orbits or a pair of homoclinic orbits (Jackson, 1992). These orbits can be further extended for small ϵ by perturbation methods and the existence of these orbits large but finite values of r can then be established (Robbins, 1979; Li & Zhang, 1993).

1.8 Use recursion relation (1.28) to compute the first terms in series (1.27). Show by direct computation that unless the first condition in (1.31) holds, one cannot compute $\mathbf{a}_2$.

Chapter 2:

Integrability: an algebraic approach

Il déterminait la forme des fonctions semblables
dont les variations sont liées entre elles par une
équation, et qui, multipliées par des facteurs constants
et ajoutés ensemble, deviennent intégrables algébriquement
bien que chacune d'elles en particulier ne le soit pas.[1]
Condorcet (1743–1794)

In this chapter, we define general notions concerning systems of differential equations, first integrals and Lax pairs. First integrals are functions of the dependent and independent variables that remain constant along the solutions of a given system. By extension, we also define the notion of second and third integrals. These are functions which remain constant on some invariant sets. If sufficiently many first integrals are known globally, the system may be completely integrated and is refer to as "integrable". Therefore, in order to define the notion of integrability, we take particular care in defining the function sets in which these first integrals belong. We give different methods to build first, second, and third integrals for polynomial vector fields and also introduce Lax pairs for systems of differential equations and describe some of their algebraic properties.

2.1 First integrals

First, we define some general notions regarding systems of differential equations and their invariants. These conserved quantities are usually called *first integrals* or *constants of the motion*.

Let $\mathbb{K}$ be a field of characteristic zero[2] (typically, we will take $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1 \dots x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. That is, $\mathbb{K}[\mathbf{x}]$ is the set of all polynomials in the variables $\mathbf{x}$ with coefficients in $\mathbb{K}$. Let $\mathbb{K}(\mathbf{x}) = \mathbb{K}(x_1 \dots x_n)$ be the quotient field of $\mathbb{K}[\mathbf{x}]$, that is, the set of all rational functions of $\mathbf{x}$ with coefficients in $\mathbb{K}$. We consider systems of differential equations that are of class C^k with $k > 0$ on an open set U of $\mathbb{K}^n$, that is, we consider a set of n functions $\mathbf{f} = (f_1, \dots, f_n) \in C^k(\mathbb{K}^n)^n$ and the system differential equations

$$\frac{dx_i}{dt} = f_i(x_1(t), \dots, x_n(t)), \qquad i = 1, \dots, n, \tag{2.1}$$

[1] "He determined the form of similar functions whose variations are related by an equation and when multiplied and summed together become algebraically integrable while every single part is not."

[2] A field with multiplication $*$ is of characteristic zero if the only element a of the field such that $a * b = 0$ for all b is $a = 0$.

or, in a more compact notation,

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{K}^n. \tag{2.2}$$

The *vector field* corresponding to the system of ODEs (2.1) is

$$\begin{aligned} \delta_{\mathbf{f}} &= \mathbf{f}.\partial_{\mathbf{x}}, \\ &= \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}. \end{aligned} \tag{2.3}$$

The correspondence between the system (2.1) and the vector field (2.3) is obtained by defining the *time derivative* of functions of $\mathbf{x}$. Let A be such a function: $A : \mathbb{K}^n \to \mathbb{K} : \mathbf{x} \to A(\mathbf{x})$, then

$$\frac{dA}{dt} \equiv \dot{A} = \delta_{\mathbf{f}}(A) = \mathbf{f}.\partial_{\mathbf{x}} A. \tag{2.4}$$

The vector field generates a *flow* φ_t that maps a subset U of $\mathbb{K}^n$ to $\mathbb{K}^n$ in such a way that a point in U follows the solution of the differential equation. That is, $\dot{\varphi}(\mathbf{x})(t) = \mathbf{f}(\varphi(\mathbf{x}))\ \forall \mathbf{x} \in U$. The time derivative is also called the *derivative along the flow* since it describes the variation of a function of $\mathbf{x}$ with respect to t as $\mathbf{x}$ evolves according to the differential system. The vector field can then be written as a system of ODEs

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2.5}$$

and the two notions, vector fields and systems of differential equations, can be used interchangeably.

Definition 2.1 A *time-independent first integral* of $\delta_{\mathbf{f}}$ on an open subset $U \in \mathbb{K}^n$ is a C^1 function $I = I(\mathbf{x}) : U \to \mathbb{K}$ such that $\delta_{\mathbf{f}} I = 0\ \forall\ \mathbf{x} \in U$. The first integral is *trivial* if $I \in \mathbb{K}$, *non-trivial* otherwise.

We want to identify functions of the variables $\mathbf{x}$ which remain constant along all solutions of a systems of ODEs. These functions, known as *first integrals* are of particular interest since the solutions lie on their level sets.

Definition 2.2 A *time-dependent first integral* of $\delta_{\mathbf{f}}$ on an open subset $U \subset \mathbb{K}^n$ and interval $T \in \mathbb{R}$ is a C^1 function $I = I(\mathbf{x}, t) : U \times T \to \mathbb{K}$ such that $\delta_{\mathbf{f}} I + \frac{\partial I}{\partial t} = 0$ with $\frac{\partial I}{\partial t} \neq 0$.

In general, we refer to a function I as a *first integral* if it is a time-dependent or time-independent non-trivial first integral. The trivial first integrals are referred to as *constants*. The set U on which the first integral is defined is the largest open subset on which the global function I remains a C^1 function and will not be explicitly given hereafter. The fact that a first integral is constant along all solutions is a direct consequence of the identification (2.4). If I is constant along the solutions, then $\frac{d}{dt}\left(I(\mathbf{x}(t), t)\right) = 0$, which is equivalent to $\partial_t I + \mathbf{f}.\partial_{\mathbf{x}} I = 0$.

Lemma 2.1 *Let $I(\mathbf{x}, t)$ be a C^1 function on an open set $U \in \mathbb{K}^n$. Then, I is a first integral of $\delta_{\mathbf{f}}$ if and only if it is constant along any solutions of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ defined on a given time interval T such that $\mathbf{x}(t) \in U\ \forall\ t \in T$. That is, $I(\mathbf{x}(t), t)$ is independent of t for all $t \in T$.*

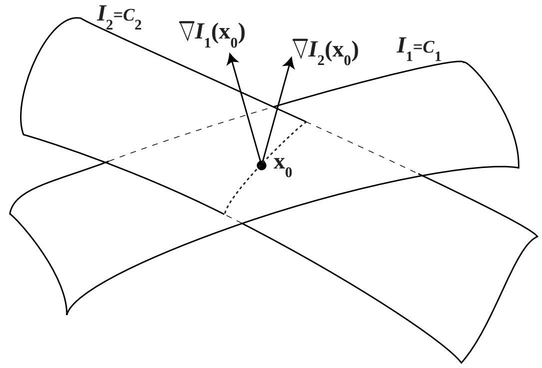

Figure 2.1: Independence of first integrals: two pieces of level sets and their intersection at $\mathbf{x} = \mathbf{x}_0$.

If I is a first integral, so is I^2 (since $\delta_{\mathbf{f}} I^2 = 2I\delta_{\mathbf{f}} I = 0$) and so is $F(I)$ where F is any C^1 function. These different first integrals are clearly not independent. Therefore, to distinguish between two different first integrals, it is useful to introduce a notion of independence based on the idea of linear independence between the gradients of first integrals. Two first integrals are *independent* if there is a point on which their gradients are linearly independent (Figure 2.1). If such a point $\mathbf{x}_0$ exists, then there is a neighborhood of $\mathbf{x}_0$ where the first integrals are independent and this domain of independence can be continued to all points where their gradients do not vanish identically. In the case of k first integrals, we use the following definition as a convenient working definition.

Definition 2.3 Let $I_1, \ldots, I_k$ be k first integrals of $\delta_{\mathbf{f}}$ defined on the subsets $U_1, \ldots, U_k$ and let $U = \bigcap_{i=1}^{k} U_i$ be a non-empty set. The first integrals form a set of *functionally independent first integrals* (or *independent first integrals*) if there exists $\mathbf{x}_0 \in U$ such that

$$\operatorname{rank}\left(\partial_{\mathbf{x}} I_1(\mathbf{x}_0), \partial_{\mathbf{x}} I_2(\mathbf{x}_0), \ldots, \partial_{\mathbf{x}} I_k(\mathbf{x}_0)\right) = k. \tag{2.6}$$

Example 2.1 The Rabinovich system. The Rabinovich system has been used as a model to describe the interaction of three quasisynchronous waves in a plasma with quadratic nonlinearities (Pikovskii & Rabinovich, 1981). It reads

$$\dot{x} = -2y^2 + \gamma x + z - \delta^2, \tag{2.7.a}$$
$$\dot{y} = 2xy + \gamma y - \delta x, \tag{2.7.b}$$
$$\dot{z} = -2zx - 2z, \tag{2.7.c}$$

where $x, y, z, \delta, \gamma \in \mathbb{R}$. The integrability of this system has been analyzed in detail (Bountis *et al.*, 1984; Giacomini *et al.*, 1991). In particular, for $\delta = 0$ and $\gamma = -1$, there exist two time-dependent first integrals (see Figure 2.2)

$$I_1 = (x^2 + y^2 + z)e^{2t}, \quad I_2 = zye^{3t}. \tag{2.8}$$

For $(x, y, z) \neq (0, 0, 0)$, we can check that the gradients of I_1 and I_2 are linearly independent. We can also verify that for $\gamma = 0$, $I = z(y - \frac{\delta}{2})e^{2t}$ is another first integral. ■

Example 2.2 A third order normal form. The following third order differential equation has been obtained as a normal form associated with a triple zero Jordan block in the study of the Kirchhoff

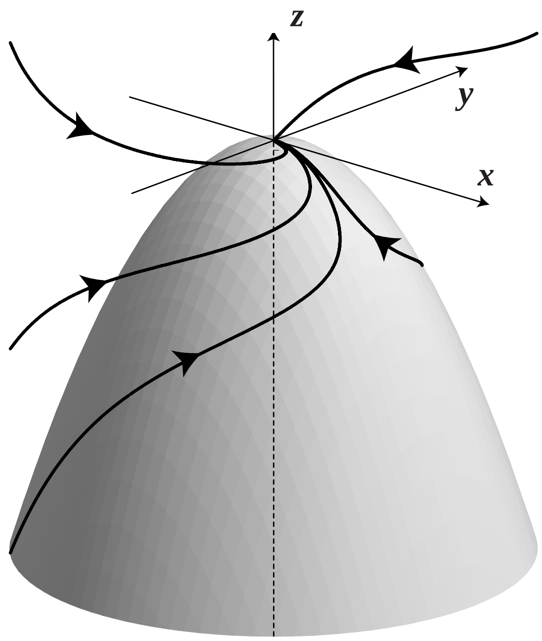

Figure 2.2: Example 2.1: as $t \to \infty$ all trajectories end up on the origin (a global attractor) by approaching the asymptotic level set $I_1 = 0$.

equations for elastic rods with intrinsic curvature (Goriely & Tabor, 1998):

$$\dddot{x} = (\dot{x})^2 \left(\alpha + x^2 - (\dot{x})^2\right), \tag{2.9}$$

where α is a free parameter. Alternatively, it can be written as the system of three first order ODEs

$$\dot{x} = y, \tag{2.10.a}$$
$$\dot{y} = z, \tag{2.10.b}$$
$$\dot{z} = y(\alpha + x^2 - y^2). \tag{2.10.c}$$

A complete description of the orbits of a third order system such as (2.10) is not possible in general. However, this system possesses some remarkable properties which allows us to obtain a complete description of the orbits. In particular, there exists a pair of first integrals (I_1, I_2) for this system given by

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \cos(\sqrt{2}x) & -\sin(\sqrt{2}x) \\ \sin(\sqrt{2}x) & \cos(\sqrt{2}x) \end{bmatrix} \begin{bmatrix} y^2 - x^2 + 1 - \alpha \\ \sqrt{2}(z - x) \end{bmatrix}. \tag{2.11}$$

These two first integrals can be used to integrate explicitly the equations of motion. Let $\Theta(x) = \psi - 1 + I_1 \cos(\sqrt{2}x) + I_2 \sin(\sqrt{2}x)$, then

$$y = \pm\sqrt{x^2 + \Theta(x)}, \qquad z = \frac{1}{2}\Theta'(x) + x, \tag{2.12}$$

where $x(t)$ solves $\dot{x} = \pm\sqrt{x^2 + \Theta(x)}$. To understand the geometry of the solutions, we consider the first integral

$$\begin{aligned} I &= I_1^2 + I_2^2 - (1-\alpha)^2, \\ &= -4xz + y^4 + 2z^2 + y^2[2(1-\alpha) - 2x^2] + x^2(2\alpha + x^2), \end{aligned} \tag{2.13}$$

and the surfaces $S_C = \{(x, y, z) \in \mathbb{R}^3 | I = C\}$. For fixed values of α, we are interested in the dynamics on the surface S_C for increasing values of C. In particular, we want to show the existence of bounded orbits, either homoclinic (an orbit connecting a fixed point to itself), heteroclinic (an orbit connecting two different fixed points), or periodic. Note that the x-axis is a line of fixed points and any orbit connecting points on this axis is either homoclinic or heteroclinic (see Figure 2.3).

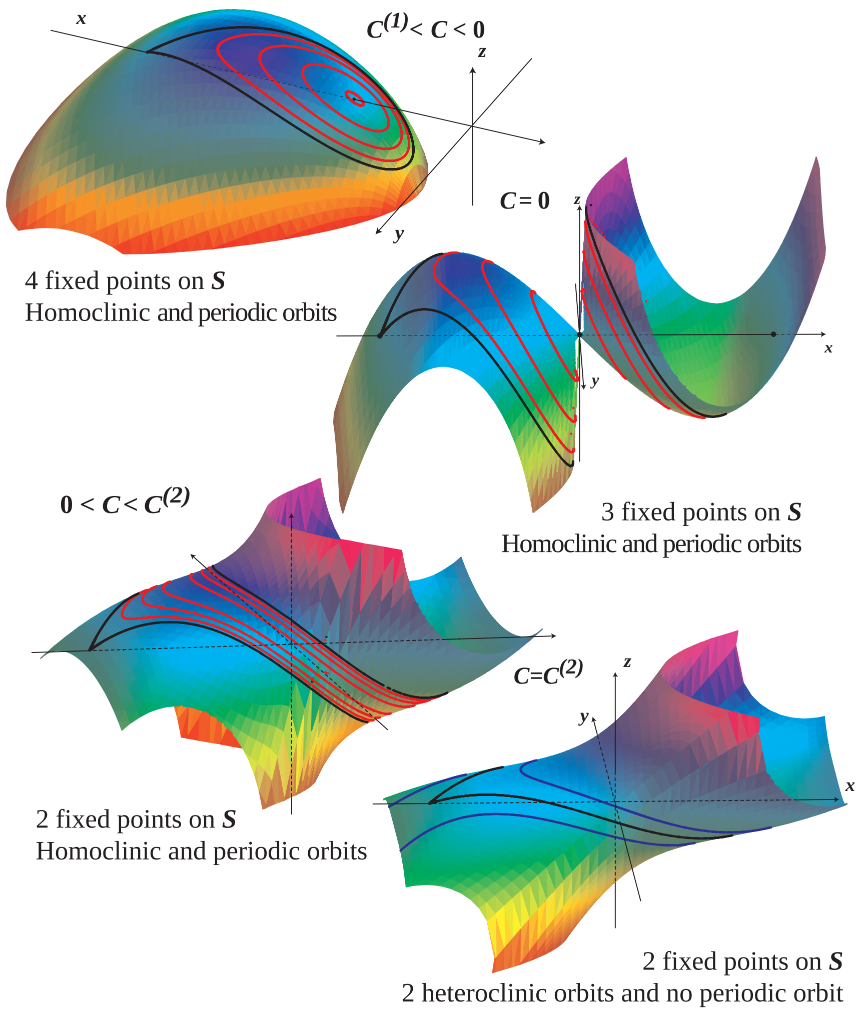

Figure 2.3: Orbits of Equations (2.10) on the level set $I(x, y, z) = C$ for various values of C.

1. For $C < C^{(1)} = -\alpha^2$, S_C has no intersection with the x-axis and there is no bounded solution.
2. For $C^{(1)} < C < 0$, there are four fixed points on S_C and there is a pair of homoclinic orbits surrounding an open set of periodic orbits.
3. For $C = 0$, two of the fixed points coalesce and there is a pair of homoclinic orbits and two families of periodic orbits.
4. For $0 < C < C^{(2)}$, there are two homoclinic orbits to the two fixed points on S_C and an open set of periodic orbits.

5. For $C = C^{(2)}$, the two homoclinic orbits collapse to a pair of heteroclinic orbits and there is no periodic orbit on $S_{C^{(2)}}$. The value $C^{(2)}$ is obtained by using the parametrization of the orbits in terms of I_1, I_2 and is found to be

$$C^{(2)} = \frac{2x_1^2}{\sin\sqrt{2}x_1} - (1-\alpha)^2, \tag{2.14}$$

where x_1 is the intersection point of $S_{C^{(2)}}$ with the x-axis, that is, the solution of

$$x_1^2 + \alpha - 1 + \sqrt{2}x_1 \cot(\sqrt{2}x_1) = 0. \tag{2.15}$$

■

2.1.1 A canonical example: The rigid body motion

The Euler equations describe the motion of a rigid body around a fixed point. These equations will be used as a paradigm throughout this book to illustrate the different concepts of integrability and nonintegrability.

Example 2.3 Euler equations. Consider a rigid body in a gravity field moving around a fixed point **O**. By a proper choice of units we set the product of the constant of gravity g and the mass m of the body to be one. Now, consider two cartesian frames, the first one fixed in space (x_1, y_1, z_1), the second one, (x, y, z), moving with the rigid body and directed along the principal axes of inertia of the body about the point **O**. The principal moments of inertia are the eigenvalues of the inertia tensor and are denoted $\mathbf{J} = (J_1, J_2, J_3)$ and we define the matrix $J = \mathrm{diag}(\mathbf{J})$. Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ be the coordinates of a unit vector along the z_1 axis and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ the coordinates of a vector of instantaneous rotation, both in the moving basis. The center of mass of the body is located in the moving frame at the point $\mathbf{X} = (X_1, X_2, X_3)$. Since $\mathbf{J}$ and $\mathbf{X}$ are fixed, the motion is determined by the two vectors $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ (see Figure 2.4). It is a standard, but by no means straightforward, matter (see for instance Goldstein's book (1980)) to show that the *Euler-Poisson equations* describing the motion of the rigid body about the fixed point are

$$J\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (J\boldsymbol{\omega}) = \mathbf{X} \times \boldsymbol{\gamma}, \tag{2.16.a}$$
$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0. \tag{2.16.b}$$

In components, it reads

$$J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3 = X_3\gamma_2 - X_2\gamma_3, \tag{2.17.a}$$
$$J_2\dot{\omega}_2 + (J_1 - J_3)\omega_3\omega_1 = X_1\gamma_3 - X_3\gamma_1, \tag{2.17.b}$$
$$J_3\dot{\omega}_3 + (J_3 - J_2)\omega_1\omega_2 = X_2\gamma_1 - X_1\gamma_2, \tag{2.17.c}$$
$$\dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3, \tag{2.17.d}$$
$$\dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1, \tag{2.17.e}$$
$$\dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2. \tag{2.17.f}$$

This system admits, for all values of the 6 parameters $J_1, J_2, J_3, X_1, X_2, X_3$, the three time-independent first integrals

$$I_1 = \boldsymbol{\gamma}.\boldsymbol{\gamma} = 1, \tag{2.18.a}$$
$$I_2 = \frac{1}{2}(J\boldsymbol{\omega}).\boldsymbol{\omega} + \mathbf{X}.\boldsymbol{\gamma}, \tag{2.18.b}$$
$$I_3 = (J\boldsymbol{\omega}).\boldsymbol{\gamma}. \tag{2.18.c}$$

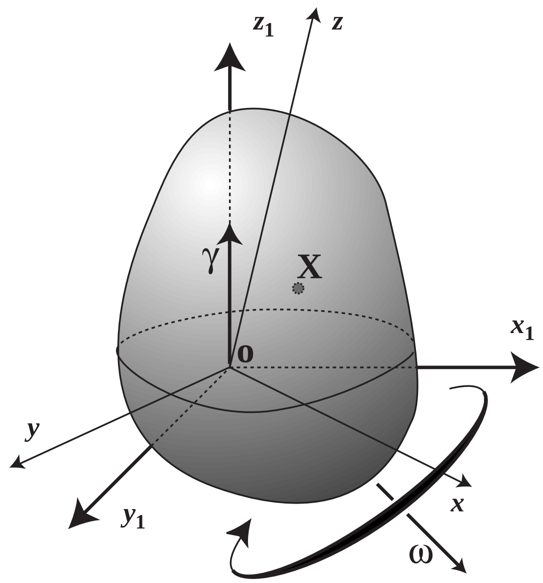

Figure 2.4: A rigid body, the frame (x, y, z) is attached to the body whose center of mass is located at the point $\mathbf{X}$. The frame (x_1, y_1, z_1) is fixed in space. The vector $\boldsymbol{\gamma}$ gives the position of the unit vector along z_1 in the moving coordinates and $\boldsymbol{\omega}$ is the angular velocity of the body (also in the moving frame).

The first relation is due to the fact that $\boldsymbol{\gamma}$ is a unit vector, the second is a form of the energy conservation and the third is the conservation of the vertical component of the angular momentum (the gravity force being vertical).

For a general n-dimensional system, we will see that $(n-1)$ first integrals are needed to integrate the system completely. However, we show in Section 2.11 that, in the problem of the rigid body motion, only four first integrals are required to complete the integration. Therefore, only one extra first integral is needed. This elusive fourth first integral was nicknamed in classical mechanics the "mathematical mermaid" (Tabor, 1989). The general system has only three first integrals (that is, for all values of the parameters $\mathbf{X}, \mathbf{J}$) and a fourth first integral has only been found in four particular cases.

1. **The complete symmetric case:** $J_1 = J_2 = J_3$. In this case, the fourth first integral is

$$I_4 = \boldsymbol{\omega}.X. \tag{2.19}$$

2. **The Euler-Poinsot case:** $X_1 = X_2 = X_3 = 0$**.** This condition implies that the center of gravity is the fixed point and the extra first integral is

$$I_4 = (\mathbf{J}\boldsymbol{\omega}).(\mathbf{J}\boldsymbol{\omega}). \tag{2.20}$$

3. **The Lagrange-Poisson case:** $X_1 = X_2 = 0$ and $J_1 = J_2$. This condition implies that the top is a symmetric top with the center of gravity along the z-axis (see Figure 2.5). In this case the angular moment along the axis of rotation is also conserved, that is,

$$I_4 = \omega_3. \tag{2.21}$$

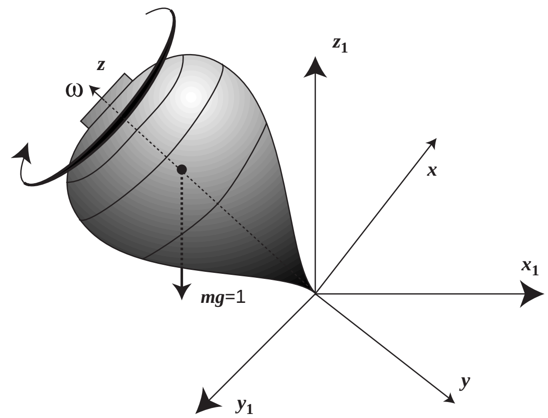

Figure 2.5: The Lagrange-Poisson case: the conserved quantity is the angular momentum around the z-axis (the axis of rotation).

4. **The Kovalevskaya case:** $J_1 = J_2 = 2J_3$ and $X_3 = 0$. The fourth first integral is

$$I_4 = \left\{(\omega_1 + i\omega_2)^2 + X_1(\gamma_1 + i\gamma_2)\right\}\left\{(\omega_1 - i\omega_2)^2 + X_1(\gamma_1 - i\gamma_2)\right\}. \tag{2.22}$$

The integrability of the Euler equations will be further analyzed in the next sections and chapters. ■

2.2 Classes of functions

First integrals can belong to different classes of functions. For instance, they can be polynomial as in the Euler equations or trigonometric as in Example 2.2. The specification of classes will eventually be crucial in demonstrating the non-existence of first integrals for a given system. Local integrability is always guaranteed (see Section 2.10.1). Therefore, the question of integrability or nonintegrability concerns the existence of invariants globally defined. If we wish to prove the nonintegrability of a given system, we must show that some analytic properties of the solutions are incompatible with the existence of a global invariant in a given class of functions (polynomial, rational, algebraic, analytic,...).

Example 2.4 The Lotka-Volterra ABC system. Depending on the values of the parameters a given system can admit different types of first integrals. The Lotka-Volterra system (Bountis *et al.*, 1984)

$$\dot{x} = x(Cy + z), \tag{2.23.a}$$
$$\dot{y} = y(x + Az), \tag{2.23.b}$$
$$\dot{z} = z(Bx + y), \tag{2.23.c}$$

where A, B, C are free parameters, is an example of such a system. Due to its rich structure, this system has become a paradigm for both integrability (Strelcyn & Wojciechowski, 1988; Grammaticos *et al.*, 1990a; Goriely, 1992) and nonintegrability (Moulin-Ollagnier, 1997; Moulin-Ollagnier, 1999; Moulin-Ollagnier & Nowicki, 1999) in low-dimensional vector fields. For different values of the parameters

(A, B, C), the system admits different types of first integrals such as

$$(A,B,C) = (-\frac{1}{2}, -\frac{1}{2}, 1), \qquad I = x^2 + y^2 + \frac{z^2}{4} - 2xy + yz + xz, \tag{2.24.a}$$

$$(A,B,C) = (1,1,1), \qquad I = \frac{(x-y)(y-z)}{y}, \tag{2.24.b}$$

$$(A,B,C) = (1,1,0), \qquad I = \frac{y}{x} + \log(1 - \frac{y}{z}), \tag{2.24.c}$$

$$(A,B,C) = (1,\sqrt{2},1), \qquad I = \frac{z(y-x)^{\sqrt{2}+1}}{xy^{\sqrt{2}}}. \tag{2.24.d}$$

These integrals are, respectively, polynomial, rational, logarithmic and transcendental functions of the variables. Moreover, if $ABC + 1 = 0$, the system admits the first integral

$$I = -AB \log|x| + B \log|y| - \log|z| + (AB+1)\log|x - Cy + Az|. \tag{2.25}$$

■

This example motivates the following definitions.

Definition 2.4 A first integral I is a *formal integral* around $\mathbf{x} = \mathbf{y}$ if it can be locally expanded in a formal power series in the form

$$I = \sum_{m_1=1}^{\infty} \dots \sum_{m_n=1}^{\infty} c_{m_1,\dots,m_n} \prod_{j=1}^{n} (x_j - y_j)^{m_j}. \tag{2.26}$$

Furthermore, I is *analytic* around $\mathbf{x} = \mathbf{y}$ if this power series is convergent in an open disk centered at $\mathbf{y}$.

These first integrals are only locally defined. To study the existence of globally defined first integrals we must specify global functions.

Definition 2.5 A first integral I is *polynomial* if $I \in \mathbb{K}[\mathbf{x}]$, it is *rational* if $I \in \mathbb{K}(\mathbf{x})$. A function $f(\mathbf{x})$ is *algebraic* over $\mathbb{K}$ if it there exist $q_0, \dots, q_s \in \mathbb{K}[\mathbf{x}]$, $s > 0$ such that

$$q_0 + q_1 f + \dots + q_s f^s = 0. \tag{2.27}$$

If s is the smallest positive integer such that (2.27) holds, the relation (2.27) is referred to as the *minimal polynomial* of f. A first integral $I(\mathbf{x}) = C$ is *algebraic* if $I(x)$ is algebraic, that is, there exist $q_0, \dots, q_s \in \mathbb{K}[\mathbf{x}]$, $s > 0$ such that

$$q_0 + q_1 C + \dots + q_s C^s = 0. \tag{2.28}$$

If $s = 1$ and $q_1 \in \mathbb{K}$ in (2.28) the first integral is polynomial, if $s = 1$ and $q_1 \notin \mathbb{K}$, it is rational. When $s = 2$ the first integral is given by the roots of a quadratic equation for C. A function is *transcendental* if it is not algebraic.

Definition 2.6 A first integral I is *logarithmic* if there exist $J_0, \dots, J_s$, $s \geq 1$ algebraic functions over $\mathbb{K}$ and $c_0, \dots, c_s \in \mathbb{K}$, such that

$$I = J_0 + \sum_{i=1}^{s} c_i \log J_i. \tag{2.29}$$

The notion of polynomial first integrals can be generalized to include sums of arbitrary *quasimonomial* terms, that is, real powers of the dependent variables.

Definition 2.7 A first integral I is *quasimonomial* if it is of the form

$$I = \sum_{i=1}^{m} a_i \prod_{j=1}^{n} x_j^{\alpha_{ij}}, \tag{2.30}$$

where $\alpha_{ij} \in \mathbb{K}$.

Note that the characterization of first integrals depends only on the dependence on $\mathbf{x}$ and not on t which is left arbitrary. For instance, in the case of a time-dependent polynomial first integral, the coefficients of the polynomial are themselves function of time.

Example 2.5 More complicated first integrals. There is yet another degree of complexity in the type of functions needed to describe first integrals (Hietarinta, 1984). For instance, the system

$$\dot{x} = y + 2uv, \tag{2.31.a}$$
$$\dot{y} = 1, \tag{2.31.b}$$
$$\dot{u} = v + 2uy, \tag{2.31.c}$$
$$\dot{v} = -2yv, \tag{2.31.d}$$

is completely integrable with Hamiltonian $I_1 = H = \frac{1}{2}(y^2+v^2)+2yuv-x$ and two extra first integrals:

$$I_2 = y^2 + \log v, \tag{2.32.a}$$
$$I_3 = -u\exp(-y^2) + \frac{\sqrt{2\pi}}{4} v \exp(y^2)\operatorname{erf}(y\sqrt{2}), \tag{2.32.b}$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)dt$ is the error function. The Hamiltonian is a polynomial first integral, I_2 is logarithmic and I_3 is transcendental. Another example also due to Hietarinta (1984) is

$$\dot{x} = y, \tag{2.33.a}$$
$$\dot{y} = -1/u, \tag{2.33.b}$$
$$\dot{u} = v, \tag{2.33.c}$$
$$\dot{v} = x/u^2. \tag{2.33.d}$$

This is again an Hamiltonian system with $I_1 = H = \frac{1}{2}(y^2 + v^2) + x/u$. However, the two extra first integrals can only be described in terms of Whittaker functions:

$$I_2 = \frac{1}{2}u\left[vW_+\left(\frac{H}{2}, y\right) + 2W_+'\left(\frac{H}{2}, y\right)\right]^2, \tag{2.34.a}$$
$$I_3 = \frac{vW_-\left(\frac{H}{2}, y\right) + 2W_-'\left(\frac{H}{2}, y\right)}{vW_+\left(\frac{H}{2}, y\right) + 2W_+'\left(\frac{H}{2}, y\right)}, \tag{2.34.b}$$

where $(\ )'$ denotes the derivative with respect to the second argument of $W(.,.)$. The two functions $W_+(a,t)$ and $W_-(a,t) = W_+(a,-t)$ are the two standard solutions of the equation

$$\ddot{x} + (\frac{1}{4}t^2 - a)x = 0, \tag{2.35}$$

chosen such that the Wronskian[3] is equal to unity. ■

2.2.1 Elementary first integrals

The concept of elementary functions comes from the work of Liouville on finite integration (Liouville, 1834). The problem is to decide whether the integral of a given function can be expressed in a "finite explicit function" (that is, using finite algebraic combinations of known functions such as log, exp, sin, cos,...). For instance, $F(x) = \int^x 2xe^{x^2} dx$ can be expressed as $F(x) = e^{x^2}$ but $\int^x e^{-x^2} dx$ has no such simple form,[4] hence the need to introduce a new function $\text{erf}(x)$. To build elementary functions of one variable, one applies a finite number of algebraic operations and the operation of taking the exponential or logarithm.

Example 2.6 Consider the function $F(x) = x^3\sqrt{x} + \exp(x \log(x + \sqrt{x})^2)$. This function can be obtained by introducing a series of new variables. First, we introduce $y_1 = \sqrt{x}$ which is algebraic in x. Second, we define $y_2 = \log(x + \sqrt{x})$ obtained by taking the logarithm of a polynomial combination of the previous variables $x + y_1$. Finally, let $y_3 = \exp(xy_2^2)$. The original function is now polynomial in the new variables $F(x) = x^3 y_1 + y_3$. Each new variable is obtained as one operation performed on a polynomial combination of the previous variables. ■

This construction can be further formalized by the concept of *elementary extension of differential fields* (see the review by Singer (1990)). Trigonometric functions and their inverse can also be obtained by algebraic combinations and exponentiations using complex numbers. As shown in the previous example, elementary functions can be multivalued and have to be restricted on some nonempty simply connected open sets.

Now, assume that $F(x) = \int f(x)dx$ can be integrated explicitly, that is, let $F(x)$ be an *elementary function*. Then, Liouville's theorem (1835) states that $F(x)$ is of the form

$$F(x) = g_0(x) + \sum_{i=1}^{m} c_i \log(g_i(x)), \tag{2.36}$$

where the c_i's are constant and the g_i's are algebraic functions of the functions appearing in $f(x)$ (see next section for a modern statement of this result). The method of simple fraction decomposition for rational functions is the simplest example of Liouville's result.

The following is a convenient working definition of elementary functions and their generalization, the Liouvillian functions

Definition 2.8 A function $F(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}^n$ is called *elementary* if it is part of the set Λ of elementary functions. The set Λ is obtained from rational functions on $\mathbb{C}^k$, $k = 0, 1, \ldots$, using a finite series of the following operations.

1. Algebraic operations: If $f_1, f_2 \in \Lambda$, then, $f_1 \star f_2 \in \Lambda$, where the operator $\star$ is either addition, substraction, multiplication and division.

2. Solution of algebraic equations: If $f_0, \ldots, f_{s-1} \in \Lambda$, then $\Psi(x) \in \Lambda$, where $\Psi(x)$ is a solution of

$$f_0 + f_1\Psi(x) + \ldots + f_{s-1}\Psi^s(x) = 0. \tag{2.37}$$

[3]The *Wronskian* of $f(x)$ and $g(x)$ is the determinant of the matrix $\begin{bmatrix} f & g \\ f' & g' \end{bmatrix}$.

[4]The proof that $\int^x e^{-x^2} dx$ is not elementary is far from being obvious (Rosenlicht, 1972) and was actually given by Liouville himself (Lützen, 1990).

3. Derivation: If $f \in \Lambda$, then $\dfrac{\partial f}{\partial x_i} \in \Lambda$.

4. Exponential and logarithm operation: If $f \in \Lambda$, then $\exp(f), \log f \in \Lambda$.

Furthermore, if we add the operation of integration: $f \in \Lambda$, then $\int f dx_i \in \Lambda$, then Λ is the set of *Liouvillian functions.*

The Liouvillian functions are generally referred to as "the functions that can be represented by quadratures" (Żołądek, 1998b). For instance, Liouville (1839; 1841) was able to prove that the Riccati equation

$$\frac{dx}{dt} = t^\alpha - x^2, \tag{2.38}$$

can be solved in terms of quadratures only for $\alpha = -2$ and $\alpha = 4k/(1-2k)$, $k \in \mathbb{N}$. For all other values, the general solution $x(t)$ is not Liouvillian.

The general algorithm to decide if the integral of an elementary function is itself elementary was given by Risch (1969; 1970) and implemented by Bronstein (1990a). See also Bronstein (1997) for a complete treatment of the problem.

2.2.2 Differential fields

In order to define elementary functions in many variables, some notions of differential algebra are required. We follow here the exposition of Schlomiuk (1993a). We start with elementary functions in one variable.

Definition 2.9 Let R be a ring. A *derivation* D is a map $D : R \to R$ such that $D(r+s) = D(r) + D(s)$ and $D(rs) = rD(r) + D(r)s$ for all $r, s \in R$.

Example 2.7 The field $\mathbb{R}[t]$ of polynomial functions in t with coefficients in $\mathbb{R}$ with the usual derivation $\frac{d}{dt}$ is a derivation. ■

Example 2.8 Take the field $R = \mathbb{R}[\mathbf{x}]$ where $\mathbf{x} = (x_1 \dots x_n)$ and pick a set of C^1 functions, $\mathbf{f}$, over $\mathbb{R}^n$. Then, the vector field $\delta_{\mathbf{f}}$ (as previously defined) is a derivation on R. Note that different sets of functions $\mathbf{f}$ define different derivations. ■

The *constants of* D are the elements of $r \in R$ such that $D(r) = 0$, and form a subring denoted $C(R, D)$. For instance, in the previous example, the set of constants of the derivation $\delta_{\mathbf{f}}$ is the set of time-independent first integrals (trivial and non-trivial).

Definition 2.10 A *differential field* is a field k together with a collection of derivation of k: $\Delta = \{D_i\}_{i \in I}$(where $I = \{1, \dots, n\}$) and is noted $\{k, \Delta\}$. An *extension* of $\{k, \Delta\}$ is a field extension K of k together with derivation $\Delta' = \{D'_i\}_{i \in I}$ such that the restriction of Δ' on k is Δ.

The set of *constants* of a differential field is the union of $C(k, D_i)$ for $i \in I$. We can define elementary monomials over k.

Definition 2.11 Let $\{K, \Delta'\}$ be a differential field extension of $\{k, \Delta\}$. The element $y \in K$ is an *elementary monomial over* k if y is transcendental over k and there exists $x \in k \setminus \{0\}$ such that for all $D'_i \in \Delta$ either

1. $D_i' y = \dfrac{D_i x}{x}$, that is, y *is logarithmic over* k and we write $y = \log(x)$ or,

2. $D_i' y = (D_i x) y$, that is, y *is exponential over* k and we write $y = \exp(x)$.

There are two ways to extend a differential field to create an elementary extension.

Definition 2.12 The extension $\{K, \Delta'\}$ of $\{k, \Delta\}$ is an *elementary extension*, if there exist $y_1, \ldots, y_m$ in K such that $K = k(y_1, \ldots, y_m)$ with either

1. y_i algebraic over $k(y_1, \ldots, y_{i-1})$ or,

2. y_i elementary over $k(y_1, \ldots, y_{i-1})$.

We can now restate Liouville's theorem on integration in finite terms (Rosenlicht, 1972).

Theorem 2.1 *Let $\{k, D\}$ be a differential field of characteristic zero, and let y be in k. If z is an element of an elementary extension $\{K, D'\}$ of k sharing the same constants and $D'z = y$, then there exist $c_1, \ldots, c_m$ and $g_0, \ldots, g_m$ in k such that*

$$y = Dg_0 + \sum_{i=1}^{m} c_i \frac{Dg_i}{g_i}. \tag{2.39}$$

In order to define elementary functions in many variables, we start with the set, $\mathbb{C}(\mathbf{x})$, of rational functions in n variables over $\mathbb{C}$ and consider extension fields with n commuting derivations $\partial_i = \partial_{x_i}$. A system of differential equations defines another derivation, the vector field $\delta_{\mathbf{f}} = \mathbf{f}.\partial_{\mathbf{x}}$ on k. One can look now for an extension of Liouville's theorem to system of differential equations. This is the content of Prelle-Singer's theorem given in Section 2.8.

2.3 Homogeneous vector fields

Consider a polynomial function $A(\mathbf{x})$ with coefficients in $\mathbb{K}$,

$$A = \sum_{\mathbf{i}, |\mathbf{i}|=1}^{m} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}, \tag{2.40}$$

where $a_{\mathbf{i}} = a_{i_1, \ldots, i_n}$, $\mathbf{x}^{\mathbf{i}} = \prod_{j=1}^{n} x_j^{i_j}$, $\mathbf{i} = (i_1, \ldots, i_n)$ is a multi-index, and $|\mathbf{i}| = \sum_{j=1}^{n} i_j$. The *degree* of A is the largest integer d such that $a_{\mathbf{i}} \neq 0$ with $|\mathbf{i}| = d$ and is denoted $\deg(A)$. The polynomial is *homogeneous* if $|\mathbf{i}| = d \ \forall \ \mathbf{i}$. The degree of a collection of polynomials $\{A_1, \ldots, A_k\}$, $A_i \in \mathbb{K}[\mathbf{x}]$, is $\deg(A_1, \ldots, A_k) = \max\{\deg(A_1), \ldots, \deg(A_k)\}$. A rational function $Q \in \mathbb{K}(\mathbf{x})$ can be written $Q = P_1/P_2$ with $P_1, P_2 \in \mathbb{K}[\mathbf{x}]$. The function Q is homogeneous if both P_1 and P_2 are homogeneous and $\deg(Q) = \deg(P_1) - \deg(P_2)$. Euler's relation connects the gradient of a polynomial to its degree.

Proposition 2.1 (Euler's relation) *Let $A \in \mathbb{K}[\mathbf{x}]$ be a homogeneous polynomial of degree d. Then, $\mathbf{x}.\partial_{\mathbf{x}} A = d\, A$, that is,*

$$x_1 \frac{\partial A}{\partial x_1} + x_2 \frac{\partial A}{\partial x_2} + \ldots + x_n \frac{\partial A}{\partial x_n} = dA. \tag{2.41}$$

Proof. Let $A = \sum a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$, then $x_j \dfrac{\partial A}{\partial x_j} = \sum_{\mathbf{i}} i_j a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$. Hence, we have

$$\sum_j x_j \frac{\partial A}{\partial x_j} = \sum_j i_j \sum_{\mathbf{i}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} = d\,A. \tag{2.42}$$

□

Example 2.9 Consider for instance the polynomial $A = 12x^2 + 6y^2 - 3xy$. Then, Euler's relation reads $x(24x - 3y) + y(12y - 3x) = 2(12x^2 + 6y^2 - 3xy) = 2A$. ■

The class of homogeneous functions can be generalized to the the class of weight-homogeneous functions. Rather than considering rational functions, we consider general C^1 functions. Let $A \in C^1(\mathbb{K}^n)$, then A is *weight-homogeneous* if there exist $d \in \mathbb{K}$ and $\mathbf{w} \in \mathbb{K}^n$ with $|\mathbf{w}| > 0$ such that

$$A(t^{w_1}x_1, t^{w_2}x_2, \ldots, t^{w_n}x_n) = t^d A(x_1, \ldots, x_n) \ \ \forall\, t \in \mathbb{K}_0. \tag{2.43}$$

The *weighted degree* d of A is denoted $\deg(A, \mathbf{w})$.

Example 2.10 The function $A = 2xy + 3z^3 - yz\sin(\frac{xz}{y})$ is weight-homogeneous. Take, for instance, the weight $\mathbf{w} = (1, 2, 1)$, then $\deg(A, \mathbf{w}) = 3$. Note that the weight of the form $\mathbf{w} = (\alpha, 2\alpha, \alpha)$ is also valid, the degree is then $\deg(A, \mathbf{w}) = 3\alpha$. ■

The generalization of Euler's theorem to weight-homogeneous functions is straightforward.

Proposition 2.2 (Generalized Euler's relation) *Let $A \in C^1(\mathbb{K}^n)$ be a weight-homogeneous function of weighted degree deg$(A, \mathbf{w}) = d$, then*

$$w_1 x_1 \frac{\partial A}{\partial x_1} + w_2 x_2 \frac{\partial A}{\partial x_2} + \ldots + w_n x_n \frac{\partial A}{\partial x_n} = dA. \tag{2.44}$$

Proof. Taking the t-derivative of (2.43), we get

$$t^{w_1-1} w_1 x_1 \frac{\partial A}{\partial x_1} + t^{w_2-1} w_2 x_2 \frac{\partial A}{\partial x_2} + \ldots + t^{w_n-1} w_n x_n \frac{\partial A}{\partial x_n} = dt^{d-1} A. \tag{2.45}$$

This last relation is valid for all $t \in \mathbb{K}_0$. In particular, it is valid for $t = 1$ where the proposition holds. □

If A is homogeneous, then $\deg(A, \mathbf{1}) = \deg(A)$. A collection of n functions $A_1, \ldots, A_k \in C^1(\mathbb{K})$ in n variables is *weight-homogeneous* if there exist $\mathbf{w}, \mathbf{a} \in \mathbb{K}^n$ such that $\deg(A_i, \mathbf{w}) = a_i \ \forall\ i$. The vector $\mathbf{a}$ is the *weighted degree* of $\mathbf{f}$ with respect to $\mathbf{w}$. A vector field $\delta_{\mathbf{f}}$ is *weight-homogeneous* if there exists a weight vector $\mathbf{w} \in \mathbb{K}^n$ such that $\mathbf{f}$ is weight-homogeneous. In the particular case when $\delta_{\mathbf{f}}$ is homogeneous, that is, f_i is homogeneous of degree d for all i, we have $\mathbf{w} = \mathbf{1}$.

2.3.1 Scale-invariant systems

There is yet another important subclass of weight-homogeneous vector fields, the *scale-invariant systems*.

Definition 2.13 A weight-homogeneous vector field $\delta_{\mathbf{f}}$ is *scale-invariant* if there exists a vector $\mathbf{w}$ such that $\deg(f_i, \mathbf{w}) = w_i - 1, \quad i = 1, \ldots, n$.

This terminology comes from the invariance of the corresponding system of differential equation under the scaling transformation

$$t \to \epsilon t, \qquad x_i \to \epsilon^{w_i} x_i, \quad i = 1, \dots, n. \tag{2.46}$$

Moreover, if the system

$$\alpha_i w_i = f_i(\boldsymbol{\alpha}), \qquad i = 1, \dots, n, \tag{2.47}$$

has a non-trivial solution ($|\boldsymbol{\alpha}| \neq 0$), then there exists a particular solution of the form[5] $\mathbf{x} = \boldsymbol{\alpha} t^{\mathbf{w}}$ (see Section 3.8). In particular, a homogeneous vector field of degree d is scale-invariant with weight $w_i = (1-d)^{-1}$. However, not all weight-homogeneous vector fields are scale-invariant. For instance, linear systems with constant coefficients are homogeneous but they are not scale-invariant. Instead, they have the invariance $\mathbf{x} \to \epsilon \mathbf{x}$ and $t \to t$. Hence, scale-invariance is a typical property of nonlinear systems.

Example 2.11 A two-degree-of-freedom Hamiltonian. The four-dimensional system

$$\dot{x}_1 = x_3, \tag{2.48.a}$$
$$\dot{x}_2 = x_4, \tag{2.48.b}$$
$$\dot{x}_3 = 2x_1 x_2^3 + \frac{3}{4} x_1^3 x_2, \tag{2.48.c}$$
$$\dot{x}_4 = 5x_2^4 + 3x_1^2 x_2^2 + \frac{3}{16} x_1^4. \tag{2.48.d}$$

is scale-invariant with $\mathbf{w} = (-2/3, -2/3, -5/3, -5/3)$. Moreover, the system has two first integrals (Ramani *et al.*, 1982; Dorizzi *et al.*, 1983):

$$I_1 = \frac{x_3^2}{2} + \frac{x_4^2}{2} - x_2^5 - x_1^2 x_2^3 - \frac{3}{16} x_1^4 x_2, \tag{2.49.a}$$
$$I_2 = -x_2 x_3^2 + x_1 x_3 x_4 - \frac{1}{32} x_1^6 - \frac{3}{8} x_1^4 x_2^2 - \frac{1}{2} x_1^2 x_2^4. \tag{2.49.b}$$

Both first integrals are weight-homogeneous with respect to the weight $\mathbf{w}$ with $\deg(I_1, \mathbf{w}) = -10/3$ and $\deg(I_2, \mathbf{w}) = -4$. There is an elegant relationship between the weighted degree of first integrals and the so-called Kovalevskaya exponents characterizing the local behavior of solutions around their scale-invariant solutions. This relationship will be discussed in Section 5.3. ■

Example 2.12 Scale-invariant solutions. The system

$$\dot{x}_1 = 4x_2 - \frac{x_1^2}{2}, \tag{2.50.a}$$
$$\dot{x}_2 = x_2 x_1 + x_1^3, \tag{2.50.b}$$

is scale-invariant with $\mathbf{w} = (-1, -2)$. The scale-invariant solution is found by substituting $x_1 = \alpha_1 t^{-1}$ and $x_2 = \alpha_2 t^{-2}$ in the system and solving for $\boldsymbol{\alpha}$. There are two such solutions given by $(\alpha_1, \alpha_2) = (2/3, -1/9)$ and $(\alpha_1, \alpha_2) = (-2/3, 2/9)$. ■

[5] Note the unusual but useful notation used in this book, $\boldsymbol{\alpha} t^{\mathbf{w}}$ is the n-dimensional vector whose components are $\alpha_i t^{w_i}$.

Example 2.13 Non-existence of scale invariant solutions. Not all scale-invariant systems have scale-invariant solutions. For instance the system

$$\dot{x}_1 = x_2(x_1^2 + x_2^2)^2, \tag{2.51.a}$$
$$\dot{x}_2 = -x_1(x_1^2 + x_2^2)^2, \tag{2.51.b}$$

is clearly homogeneous of degree $d = 5$ and hence scale-invariant with $\mathbf{w} = (-1/4, -1/4)$. However, there is no scale-invariant solution since the ansatz $\mathbf{x} = \boldsymbol{\alpha} t^{-\frac{1}{4}}$ implies $\alpha_1 = \alpha_2 = 0$. ■

2.3.2 Homogeneous and weight-homogeneous decompositions

Most vector fields are neither homogeneous nor weight-homogeneous. For instance if the diagonal terms of the linear part of a nonlinear vector field do not vanish identically, then the system is not weight-homogeneous. Nevertheless, any analytic vector field can be decomposed into *homogeneous components* (that is, the power series expansion of the vector field). In the same way, many nonlinear vector fields (for instance all polynomial vector fields) can be decomposed into *weight-homogeneous components* with respect to a given weight vector $\mathbf{w} \in \mathbb{K}^n$:

$$\mathbf{f} = \mathbf{f}^{(0)} + \mathbf{f}^{(1)} + \ldots + \mathbf{f}^{(p)}, \tag{2.52}$$

where

$$f_i^{(j)}(t^{\mathbf{w}}\mathbf{x}) = t^{w_i - j - 1} f_i^{(j)}(\mathbf{x}), \qquad i = 1, \ldots, n; \quad j = 1, \ldots, p \;\forall\; t \in \mathbb{K}. \tag{2.53}$$

That is, the components $\mathbf{f}^{(j)}$ are weight-homogeneous with respect to $\mathbf{w}$ with weighted degrees $\deg(f_i^{(j)}, \mathbf{w}) = w_i - j - 1$. Also, in general, this *weight-homogeneous decomposition* is not unique and different weight vectors can be found. The corresponding decomposition on the vector field is denoted

$$\delta_{\mathbf{f}} = \delta_0 + \delta_1 + \ldots + \delta_p. \tag{2.54}$$

That is, $\delta_i = \delta_{\mathbf{f}^{(i)}}$, $i = 0, \ldots, p$. In Section 4.5, a geometric interpretation of the decomposition into weight-homogeneous components is given in terms of Newton's polyhedra in the space of exponents.

Example 2.14 A cubic planar system. Consider the vector field

$$\mathbf{f} = \begin{bmatrix} \lambda_1 x_1 + x_1(a_1 x_1 + a_2 x_1 x_2 + a_3 x_2) \\ \lambda_2 x_2 + x_2(b_1 x_1 + b_2 x_1 x_2 + b_3 x_2) \end{bmatrix}. \tag{2.55}$$

The homogeneous decomposition is just the Taylor expansion of the vector field:

$$\mathbf{f}^{(0)} = \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{bmatrix}, \quad \mathbf{f}^{(1)} = \begin{bmatrix} x_1(a_1 x_1 + a_3 x_2) \\ x_2(b_1 x_1 + b_3 x_2) \end{bmatrix}, \quad \mathbf{f}^{(2)} = \begin{bmatrix} a_2 x_1^2 x_2 \\ b_2 x_1 x_2^2 \end{bmatrix}. \tag{2.56}$$

The weight-homogeneous decomposition can be performed in three different ways:

1. $\mathbf{w} = (-1, 0), \quad \mathbf{f}^{(0)} = \begin{bmatrix} a_1 x_1^2 + a_2 x_1^2 x_2 \\ b_1 x_1 x_2 + b_2 x_1 x_2^2 \end{bmatrix}, \quad \mathbf{f}^{(1)} = \mathbf{f} - \mathbf{f}^{(0)}.$

2. $\mathbf{w} = (0, -1), \quad \mathbf{f}^{(0)} = \begin{bmatrix} a_1 x_1 x_2 + a_2 x_1^2 x_2 \\ b_1 x_2^2 + b_2 x_2^2 x_1 \end{bmatrix}, \quad \mathbf{f}^{(1)} = \mathbf{f} - \mathbf{f}^{(0)}.$

3. $\mathbf{w} = (-\frac{1}{2}, -\frac{1}{2})$, $\quad \mathbf{f}^{(0)} = \begin{bmatrix} a_2 x_1^2 x_2 \\ b_2 x_2 x_1^2 \end{bmatrix}$, $\mathbf{f}^{(1)} = \begin{bmatrix} a_1 x_1^2 + a_3 x_1 x_2 \\ b_1 x_1 x_2 + b_3 x_2^2 \end{bmatrix}$.

Note that the first term in the weight-homogeneous decomposition includes the most nonlinear terms of the vector field. Each of these decompositions can be used in the analysis of this system. ■

2.3.3 Weight-homogeneous decompositions

The property that some vector fields can be decomposed into weight-homogeneous components has an interesting consequence on the structure of the first integrals. Consider a vector field $\delta_{\mathbf{f}}$ and assumed that it can be decomposed into weight-homogeneous components. Let $\mathbf{w} \in \mathbb{K}^n$ be the weight vector of its highest component $\mathbf{f}^{(0)}$. Now, consider a function $A(\mathbf{x})$ and assume that it can also be decomposed into weight-homogeneous components with respect to the same weight vector $\mathbf{w}$, that is,

$$A = A^{(0)} + A^{(1)} + A^{(2)} + \ldots + A^{(q)}, \tag{2.57}$$

where, $\deg(A^{(0)}, \mathbf{w}) > \deg(A^{(1)}, \mathbf{w}) > \ldots > \deg(A^{(q)}, \mathbf{w})$. The highest and lowest components of a first integral of a given vector field are first integrals of, respectively, the highest and lowest components of the vector field.

Proposition 2.3 *Let $I(\mathbf{x})$ be a first integral of $\delta_{\mathbf{f}}$. Assume that there is a weight-homogeneous decomposition of $\delta_{\mathbf{f}} = \delta_0 + \delta_1 + \ldots + \delta_p$ and a decomposition of $I = I^{(0)} + I^{(1)} + \ldots + I^{(q)}$. Then, $I^{(0)}$ is a first integral of δ_0 and $I^{(q)}$ is a first integral of δ_p.*

Proof. Let $A = \delta_{\mathbf{f}} I$, and consider its weight-homogeneous components:

$$A = A^{(0)} + A^{(1)} + A^{(2)} + \ldots A^{(r)}. \tag{2.58}$$

Since I is a first integral, $A \equiv 0$ and every weight-homogeneous component must vanish identically. This gives

$$A^{(0)} = \delta_0 I^{(0)}, \tag{2.59.a}$$
$$A^{(1)} = \delta_0 I^{(1)} + \delta_1 I^{(0)}, \tag{2.59.b}$$
$$\vdots$$
$$A^{(r)} = \delta_p I^{(q)}, \tag{2.59.c}$$

and the result follows. □

Example 2.15 The Klapper-Rado-Tabor model. The following system has been used to model ideal three-dimensional incompressible magnetohydrodynamics (Klapper *et al.*, 1996):

$$\dot{x}_1 = x_4, \tag{2.60.a}$$
$$\dot{x}_2 = x_5, \tag{2.60.b}$$
$$\dot{x}_3 = x_6, \tag{2.60.c}$$
$$\dot{x}_4 = \frac{1}{3} x_1 x_3 - \frac{\beta}{3} x_1, \tag{2.60.d}$$
$$\dot{x}_5 = \frac{1}{3} x_2 x_3 + \frac{\beta}{3} x_2 - \frac{\gamma}{3} x_1, \tag{2.60.e}$$
$$\dot{x}_6 = x_3^2 + 8 x_5 - \beta^2, \tag{2.60.f}$$

where β, γ are parameters. The system admits one polynomial first integral found by using the procedure described in Section 2.4.1

$$I = 96x_2^2 + 72x_5x_3 + 2x_3^3 - 24x_2x_6 - 3x_6^2 - 72\beta x_5 - 6\beta^2 x_3 + 72\gamma x_4. \tag{2.61}$$

There are several possible choices of weight vectors. For instance, consider the vector $\mathbf{w} = (5, 3, 2, 6, 4, 3)$. The corresponding decomposition is

$$\mathbf{f}^{(0)} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ \frac{1}{3}x_1x_3 \\ \frac{1}{3}x_2x_3 - \frac{\gamma}{3}x_1 \\ x_3^2 + 8x_5 \end{bmatrix}, \quad \mathbf{f}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{\beta}{3}x_1 \\ \frac{\beta}{3}x_2 \\ 0 \end{bmatrix}, \quad \mathbf{f}^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\beta^2 \end{bmatrix}. \tag{2.62}$$

In turn, the first integral can be written $I = I^{(0)} + I^{(2)} + I^{(4)}$ where $\deg(I^{(i)}, \mathbf{w}) = 6 - i$,

$$I^{(0)} = 96x_2^2 + 72x_5x_3 - 24x_2x_6 - 3x_6^2 + 72\gamma x_4, \tag{2.63.a}$$
$$I^{(2)} = 2x_3^3 - 72\beta x_5, \tag{2.63.b}$$
$$I^{(4)} = -6\beta^2 x_3, \tag{2.63.c}$$

and it can be checked that $\delta_0 I^{(0)} = \delta_0 I^{(2)} + \delta_2 I^{(0)} = \delta_0 I^{(4)} + \delta_2 I^{(2)} + \delta_4 I^{(0)} = 0$ and $\delta_2 I^{(4)} + \delta_4 I^{(2)} = \delta_4 I^{(4)} = 0$. ■

2.4 Building first integrals

The computation for a polynomial or rational first integrals of a given degree can be performed in a straightforward way. The idea is actually quite simple but turns out to be computationally cumbersome.

2.4.1 A simple algorithm for polynomial first integrals

Consider an n-dimensional polynomial vector field $\mathbf{f}$ of degree d. In general, this vector field may depend on a certain number of parameters, say $(\mu_1, \ldots, \mu_p)$. The problem consists of finding the values of $(\mu_1, \ldots, \mu_p)$ such that the vector field admits a time-independent polynomial first integral of a given degree D.

1. Start with $D = 1$.

2. Consider the most general form of polynomial first integral of degree D

$$I(\mathbf{x}) = \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{|\mathbf{i}|=D} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}. \tag{2.64}$$

3. Compute the time derivative of $I(\mathbf{x})$:

$$\begin{aligned}
\delta_{\mathbf{f}} I &= \mathbf{f}.\partial_{\mathbf{x}} \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{|\mathbf{i}|=D} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}, \\
&= \sum_{j=1}^{n} f_j \left(\sum_{\mathbf{i},\ |\mathbf{i}|=1}^{|\mathbf{i}|=D} c_{\mathbf{i}} i_j \frac{\mathbf{x}^{\mathbf{i}}}{x_j} \right), \\
&= \sum_{\mathbf{i},\ |\mathbf{i}|=0}^{|\mathbf{i}|=D+d-1} K_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}. \qquad (2.65)
\end{aligned}$$

4. Since we are looking for I such that $\delta_{\mathbf{f}} I = 0$, we have $K_{\mathbf{i}} = 0$. This system of equations is a linear system for the coefficients $c_{\mathbf{i}}$ of dimension at most[6] $\begin{pmatrix} n+d+D-1 \\ n \end{pmatrix}$. So, if there exist values of the parameter $(\mu_1, \dots, \mu_p)$ and a set of constants $c_{\mathbf{i}}$ that are not all zero, such that $K_{\mathbf{i}} = 0$ for all $\mathbf{i}$, then $I(\mathbf{x})$ is a first integral. Otherwise increase D by 1 and return to Step 2.

In Section 4.10, we show how to implement this algorithm and illustrate it on simple examples. The construction of polynomial first integrals along these lines can easily be implemented in symbolic computation (Codutti, 1992). To simplify the computation, we can use Proposition 2.3 and decompose the computation by taking into account the weight of each component of the vector field. To do so, we compute the first integrals of the weight-homogeneous components of highest degree and then compute successively the lower components.

There are three main problems with such an analysis: first, as D increases the computation becomes extremely cumbersome. This is due to the fact that the number of coefficients of a polynomial of degree D with n variables increases as the number of integer grid points in a hypersphere of radius D in n dimensions. For instance, in dimension 5, a general polynomial of degree 4 has 126 coefficients, while a polynomial of degree 20 has 53130 coefficients. To find a non-trivial solution, an overdetermined linear system for the coefficients $c_{\mathbf{i}}$ has to be solved. Needless to say, this task becomes increasingly tedious and becomes rapidly a futile exercise in computation. For instance, polynomial integrals for the Lorenz system (the quadratic system of dimension 3 defined in Chapter 1) have been computed all the way to degree 20 (Schwarz, 1991).

Second, why restrict our attention to polynomial first integrals? We could compute as well rational or algebraic first integral. This can be done by choosing a proper ansatz and solving the corresponding equation for the coefficients. However, more analysis is needed to determine whether such a computation is really necessary or it can be simplified.

Third, is there a bound on the degree of polynomial (or algebraic) first integrals? The main problem here is that if I is a first integral, so is I^n. Therefore, the first integral I reappears in the computation as we increase the degree. The problem of finding a relationship between the degree of first integrals and some quantities that can be algorithmically computed will be thoroughly addressed in Chapter 5.

A clever way to deal with some of these problems is to show that first integrals can be obtained from simpler, more general, building blocks, the so-called *second integrals*.

[6] The symbol $\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!}$ is the usual *binomial coefficient*.

2.5 Second integrals

A first integral defines a global function in $\mathbb{K}^n$ constant under the evolution of the vector field. In particular, any constant value of the first integral allows us to define hypersurfaces invariant under the flow. Rather than considering functions that are globally constant (that is, for arbitrary values of the arbitrary parameter), we can consider functions that are only constant on a specific level set.

Example 2.16 A two-dimensional Lotka-Volterra system. Consider the system (Collins, 1996)

$$\dot{x} = x(3 - x - 5y), \tag{2.66.a}$$
$$\dot{y} = y(-1 + x + y). \tag{2.66.b}$$

It is clear that $J_1 = x$ satisfies the condition $\dot{x} = 0$ whenever $x = 0$. In the same way, $J_2 = y$ is such that $\dot{y} = 0$ when $y = 0$. A third *second integral* can be formed by taking $J_3 = x + 3y - 3$ and we verify that

$$\dot{J}_3 = J_3(y - x). \tag{2.67}$$

Therefore, on the line $x + 3y - 3 = 0$, we have $\dot{J}_3 = 0$. Moreover, the system admits a first integral obtained by taking products of the three second integrals, that is,

$$I = J_1 J_2^3 J_3^2. \tag{2.68}$$

The dynamics of the system can be obtained by the level set of the first integral in the phase plane (x, y) (see Figure 2.6). Note that the three second integrals split the phase portrait into regions where the behavior is qualitatively different (bounded or unbounded). ■

Definition 2.14 A *second integral* of a vector field $\delta_{\mathbf{f}}$ is a C^1 function $J = J(\mathbf{x}) : \mathbb{K}^n \to \mathbb{K}$ such that $\delta_{\mathbf{f}}(J) = \alpha J$ where $\alpha = \alpha(\mathbf{x}) : \mathbb{K}^n \to \mathbb{K}$. It is *trivial* if $J \in \mathbb{K}$ and is *nontrivial* otherwise.

Second integrals reduce to first integrals when $\alpha = 0$ and to time-dependent first integrals when $\alpha \in \mathbb{K}$. Second integrals were studied by Darboux, Poincaré and Painlevé among others (Darboux, 1878). They can be found in the literature under different names such as: *special integrals* (Albrecht *et al.*, 1996); *eigenpolynomials* (Astrelyn, 1991; Man, 1993); *Darboux polynomials* (Moulin-Ollagnier *et al.*, 1995); *algebraic invariant curves* (Hewitt, 1991; Kooij & Christopher, 1993; Collins, 1996; Schlomiuk, 1993a; Christopher, 1994; Collins, 1994; Żołądek, 1998a); *particular algebraic solutions* (Jouanolou, 1979); *algebraic particular integrals* (Schlomiuk, 1993b); *special polynomials* (Bronstein, 1990b); *Darboux curves* (Maciejewski & Strelcyn, 1995); *algebraic invariant manifolds* (Labrunie & Conte, 1996; Figueiredo *et al.*, 1998); or *Stationary solutions* (Burov & Karapetyan, 1990). The polynomial second integrals for polynomial vector fields play a particularly important role in the study of integrability. We will refer to these integrals as *Darboux polynomials*.

Example 2.17 The Hess-Appelrot top. Consider again the Euler equations for the rigid body motion around a fixed point (Equations (2.16))

$$J\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (J\boldsymbol{\omega}) = \mathbf{X} \times \boldsymbol{\gamma}, \tag{2.69.a}$$
$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0. \tag{2.69.b}$$

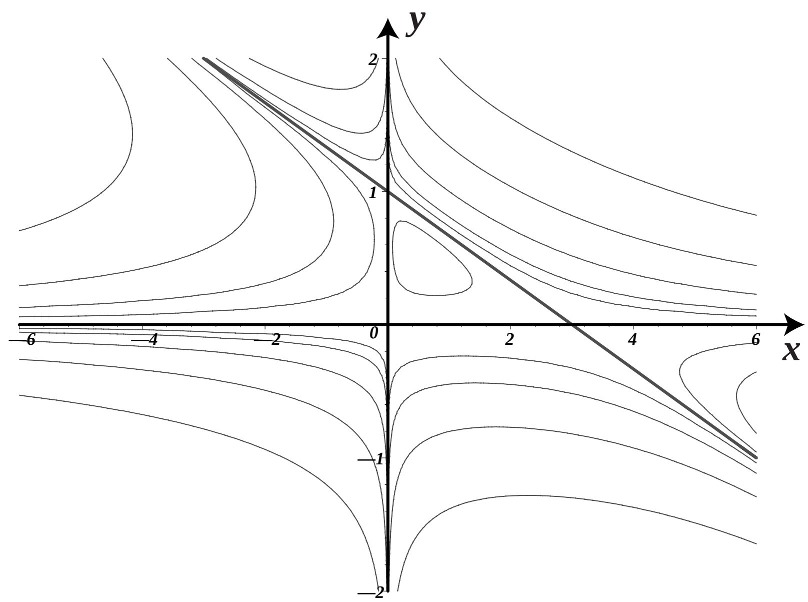

Figure 2.6: The level set of the first integral $I = J_1 J_2^3 J_3^2$ for Equations (2.66). Note that the triangle formed by the intersection of the three second integrals J_1, J_2, J_3 (in bold) contains periodic orbits.

A particular case of a second integral was found by Hess (1890) and Appelrot (1892) (Leimanis, 1965, p. 60) when $X_2 = 0$ and $\sqrt{J_1(J_2 - J_3)}X_1 + \sqrt{J_3(J_1 - J_2)}X_3 = 0$. Then, the second integral reads

$$K = J_1\omega_1 X_1 + J_3\omega_3 X_3. \tag{2.70}$$

That is, we have $\dot{K} = \alpha K$ where $\alpha = -\omega_2\sqrt{(J_1 - J_2)(J_2 - J_3)}/\sqrt{J_1 J_3}$. For a detailed analysis of the Hess-Appelrot top, see the papers by Varkhalev and Gorr (1984) and Dovbysh (1992). In particular, the relationship between the existence of separatrices in phase space and single-valued solutions in complex time will be addressed further in Chapter 7 (Ziglin, 1983a).

■

2.5.1 Darboux polynomials

The computation of a rational first integral $I = P/Q$ for polynomial vector fields is equivalent to the computation of Darboux polynomials.

Proposition 2.4 *Let $P, Q \in \mathbb{K}[\mathbf{x}]$ be two relatively prime polynomials.*[7] *Then, $I = P/Q$ is a rational first integral of $\delta_\mathbf{f}$ if and only if there exists $\alpha \in \mathbb{K}[\mathbf{x}]$ such that*

$$\delta_\mathbf{f}(P) = \alpha P \ \textit{and} \ \ \delta_\mathbf{f}(Q) = \alpha Q. \tag{2.71}$$

Proof. Let $I = P/Q$ be a first integral of $\delta_\mathbf{f}$, then $(\delta_\mathbf{f} P)Q = (\delta_\mathbf{f} Q)P$. Therefore, P divides $(\delta_\mathbf{f} P)Q$, but since P and Q are relatively prime, $\delta_\mathbf{f} P = \alpha P$. Conversely, if $\delta_\mathbf{f}(P) = \alpha P$ and $\delta_\mathbf{f}(Q) = \alpha Q$, then $\delta_\mathbf{f}(P)Q - \delta_\mathbf{f}(Q)P = \alpha PQ - \alpha QP = 0$ and P/Q is a first integral. □

[7] Two polynomials P and Q are *relatively prime* if they do not have any common factors.

The following proposition give some elementary properties of the Darboux polynomials.

Proposition 2.5 *Let P, P_1, and P_2 be Darboux polynomials, then (i) P_1P_2 is a Darboux polynomial, and (ii) all irreducible factors of P are Darboux polynomials.*

Proof. *(i)* Compute $\delta_{\mathbf{f}}(P_1P_2) = P_1\delta_{\mathbf{f}}P_2 + P_2\delta_{\mathbf{f}}P_1 = \alpha_1 P_1P_2 + \alpha_2 P_1P_2$. Hence, we have: $\delta_{\mathbf{f}}(P_1P_2) = (\alpha_1 + \alpha_2)P_1P_2$.

(ii) Let $P = Q_1^s Q_2$ with Q_1, Q_2 relatively prime polynomials and Q_1 irreducible. Then,

$$\delta_{\mathbf{f}}P = sQ_1^{s-1}\delta_{\mathbf{f}}(Q_1)Q_2 + Q_1^s\delta_{\mathbf{f}}(Q_2) = \alpha Q_1^s Q_2. \tag{2.72}$$

Since Q_1^s divides $sQ_1^{s-1}\delta_{\mathbf{f}}(Q_1)Q_2+Q_1^s\delta_{\mathbf{f}}(Q_2)$ and Q_1 is relatively prime with Q_2, Q_1 must divide $\delta_{\mathbf{f}}(Q_1)$. Also, $\delta_{\mathbf{f}}Q_2 = \alpha_2 Q_2$. By induction on Q_2, all irreducible factors of P are Darboux polynomials. □

The existence of sufficiently many Darboux polynomials implies the existence of a first integral.

Theorem 2.2 (Darboux) *Let $\mathbf{f} \in \mathbb{C}[\mathbf{x}]^n$ be a polynomial vector field of degree d and assume that $\delta_{\mathbf{f}}$ admits q Darboux polynomials $(J_i, \dots, J_q)$. Then, if $q > \binom{n+d-1}{n}$ the system admits a first integral of the form*

$$I = \prod_{i=1}^{q} J_i^{\lambda_i}. \tag{2.73}$$

where $\lambda_i \in \mathbb{C}\,\forall\, i$. Moreover, $q > \binom{n+d-1}{n} + n$ if and only if the system admits a rational first integrals (that is, $\lambda_i \in \mathbb{Z}\,\forall\, i$).

Proof. Assume that $\delta_{\mathbf{f}}$ admits q Darboux polynomials $(J_1, \dots, J_q)$, such that $\delta_{\mathbf{f}}J_i = \alpha_i J_i$ with $\deg(\alpha_i) \leq d-1$. The polynomials of degree less than or equal to $d-1$ in n variables form a complex vector space of dimension $\binom{n+d-1}{n}$. Hence, there exist $(\lambda_1, \dots, \lambda_q)$ with $\lambda_i \in \mathbb{C}$ such that $\sum_{i=1}^q \lambda_i\alpha_i = 0$ and from Proposition 2.5 it follows that $\delta_{\mathbf{f}}\prod_{i=1}^q J_i^{\lambda_i} = (\sum_{j=1}^q \lambda_j\alpha_j)\prod_{i=1}^q J_i^{\lambda_i}$. Hence, $I = \prod_{i=1}^q J_i^{\lambda_i}$ is a first integral. Moreover, as shown by Weil (1995), if $q > \binom{n+d-1}{n} + n$, we can choose $\lambda_i \in \mathbb{Z}\,\forall\, i$ and obtain a rational first integral. □

Example 2.18 Consider again Example 2.16. The three second integrals $J_1 = x$, $J_2 = y$ and $J_3 = x + 3y - 3$ have factors $\alpha_1 = 3 - x - 5y$, $\alpha_2 = -1 + x + y$ and $\alpha_3 = y - x$. Now, the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$ can be solved for all (x, y) by taking $c_1 = 1$, $c_2 = 3$, and $c_3 = 2$. Therefore, according to (2.73), the combination $I = J_1J_2^3J_3^2$ is a first integral. Indeed, we can verify that $\delta_{\mathbf{f}}I = (\alpha_1 + 3\alpha_2 + 2\alpha_3)I = 0$. ■

2.5.2 Darboux polynomials for planar vector fields

The Darboux polynomials can be used to describe completely the elementary first integrals of vector fields (see Singer (1992) for a proof).

Theorem 2.3 (Darboux) *Let $f_1, f_2 \in \mathbb{K}[x_1, x_2]$ and $\delta_{\mathbf{f}} = f_1\partial_{x_1} + f_2\partial_{x_2}$ with $d = deg(f_1, f_2)$. Let $J_1, \dots, J_r$ be Darboux polynomials, then if $r \geq 2 + \frac{d(d+1)}{2}$, there exist integers $n_1, \dots, n_r$ such that*

$$I = \prod_{i=1}^{r} J_i^{n_i} \tag{2.74}$$

is a first integral of $\delta_{\mathbf{f}}$. Moreover, if J is a Darboux polynomial, then either J divides the greatest common divisors of f_1 and f_2 or $\exists\ c_1, c_2 \in \mathbb{K}_0$ such that J divides

$$c_1 \prod_{n_i>0} J_i^{n_i} - c_2 \prod_{n_i<0} J_i^{-n_i}. \tag{2.75}$$

This proposition shows that if we know sufficiently many Darboux polynomials we can build a first integral for the system. There are two important corollaries to this theorem.

Corollary 2.1 *Let $\delta_{\mathbf{f}}$ be the vector field of Theorem 2.3. Then, either there exist both a non-trivial rational first integral and an infinity of irreducible Darboux polynomials or there exists a finite number of irreducible Darboux polynomials.*

Corollary 2.2 *There exists a positive integer N (depending on $\delta_{\mathbf{f}}$) greater than the degree of all Darboux polynomials.*

This last corollary implies that the degree of Darboux polynomials for planar vector fields is bounded. This bound may depend on the parameters of $\delta_{\mathbf{f}}$. For instance, the polynomials $J = y - x^m$ are Darboux polynomials for the system (Collins, 1996)

$$\dot{x} = x, \tag{2.76.a}$$

$$\dot{y} = my, \qquad m \in \mathbb{Z}. \tag{2.76.b}$$

The degree of J depends on the coefficient m. One problem in computing Darboux polynomials is that this bound is not known a priori. Moreover, there is, to date, no generalization of this result to higher dimensions. It is usually conjectured that such a bound exists in general. Nevertheless, there exist systematic ways to construct elementary first integrals for planar vector fields. The method outlined below rests on the following result.

Proposition 2.6 *If the planar vector field $\delta_{\mathbf{f}}$ has an elementary first integral, then there exists an integer n and a Darboux polynomial J such that*

$$\delta_{\mathbf{f}} J = -n \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) J. \tag{2.77}$$

The existence of such a Darboux polynomial provides an integrating factor for the first integral and can be used in the following algorithm due to Prelle and Singer (1983) (Shtokhamer *et al.*, 1986; Shtokhamer, 1988) and implemented by Man and MacCallum (1997) (Rocha Filho *et al.*, 1999).

2.5.3 The Prelle-Singer Algorithm

1. Let $N = 1$.

2. Find all the Darboux polynomials J_i with $\delta_{\mathbf{f}} J_i = \alpha_i J_i$ such that $\deg(J_i) \leq N$.

3. Decide if there exists a set of constants $\lambda_i \in \mathbb{K}$, not all zero, such that

$$\sum_{i=1}^{m} \lambda_i \alpha_i = 0. \tag{2.78}$$

If such λ_i's exist, then $I = \prod_{i=1}^{m} J_i^{\lambda_i}$ is a first integral. Otherwise go to Step 4.

4. Decide if there exists a set of constant $\lambda_1, \dots, \lambda_m \in \mathbb{K}$, not all zero, such that

$$\sum_{i=1}^{m} \alpha_i \lambda_i = -\left(\partial_{x_1} f_1 + \partial_{x_2} f_2\right). \tag{2.79}$$

Then, $R = \prod_{i=1}^{m} J_i^{c_i}$ is an *integrating factor* and an elementary first integral can be obtained by integrating the equations

$$\frac{\partial I}{\partial x_1} = R f_2, \quad \frac{\partial I}{\partial x_2} = -R f_1. \tag{2.80}$$

If there is no such R, and if N is less than a preset bound then increase N by 1 and return to Step 2.

Example 2.19 A simple system. Consider the system (Man & MacCallum, 1997)

$$\dot{x} = 2x^2 y - x, \tag{2.81.a}$$
$$\dot{y} = 2xy^2 + y. \tag{2.81.b}$$

For $N = 1$, we find two Darboux polynomials $J_1 = x$ and $J_2 = y$ with $\alpha_1 = 2xy - 1$, $\alpha_2 = 2xy + 1$. Moreover, we verify that if $\lambda_1 = \lambda_2 = -2$, then

$$\lambda_1 \alpha_1 + \lambda_2 \alpha_2 = -\left(\partial_{x_1} f_1 + \partial_{x_2} f_2\right). \tag{2.82}$$

Hence, $R = x^{-2} y^{-2}$ is an integrating factor and

$$I = 2 \log \frac{x}{y} - \frac{1}{xy}, \tag{2.83}$$

is a logarithmic first integral for the system. ■

Example 2.20 The planar vector field

$$\dot{x} = 3(x^2 - 4), \tag{2.84.a}$$
$$\dot{y} = 3 + xy - y^2, \tag{2.84.b}$$

has been studied in relation to Darboux-Schwartz-Christoffel integrals by Żołądek (1998a). There exist one quadratic and two quartic Darboux polynomials:

$$\begin{aligned} J_1 &= y^4 - 6y^2 - 4xy - 3, & \alpha_1 &= 4(x - y), \\ J_2 &= y^4 + 2xy^3 + 6y^2 + 2xy + x^2 - 3, & \alpha_2 &= 2(3x - 2y), \\ J_3 &= x^2 - 4, & \alpha_3 &= 6x. \end{aligned} \tag{2.85}$$

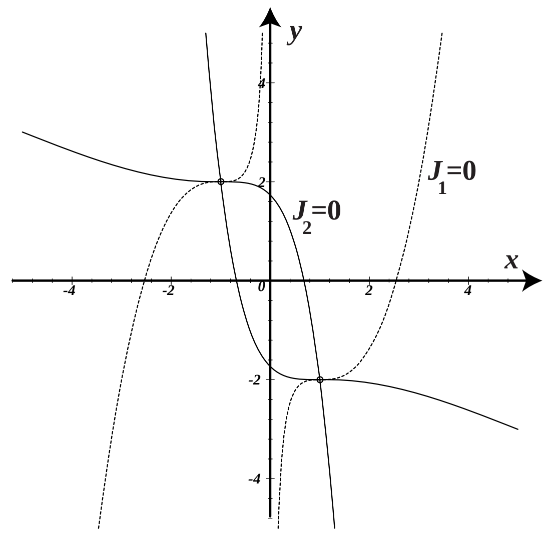

Figure 2.7: The zero level sets of the Darboux polynomial J_1 and J_2 of Example 2.20.

The level sets $J_1 = J_2 = 0$ are shown in Figure 2.7. The relationship $3\alpha_1 - 3\alpha_2 + \alpha_3 = 0$ can be used to build the rational first integral

$$I = J_3 J_1^3 J_2^{-3}. \tag{2.86}$$

Note that the fixed points $(\pm 2, \mp 1)$ are resonant, that is, the linear eigenvalues are in a 3:1 ratio. The general relationship between the existence of first integrals and resonances between linear eigenvalues is given in Chapter 5. ■

The most involved step in the Prelle-Singer algorithm is the computation of Darboux polynomials. This step can be simplified for homogeneous vector fields by using a lemma due to Collins (1996).

Lemma 2.2 (Collins) *Let $\delta_{\mathbf{f}} = f_1\partial_{x_1} + f_2\partial_{x_2}$ be a homogeneous vector field. If $W = x_1 f_2 - x_2 f_1$ does not vanish identically, then W is a Darboux polynomial for $\delta_{\mathbf{f}}$ and all irreducible homogeneous Darboux polynomials of $\delta_{\mathbf{f}}$ divide W.*

Proof. Following Weil (1995), we let $n = \deg(f_1, f_2)$. Since, f_1 and f_2 satisfy Euler's relation (Proposition 2.1), we have: $x_1\partial_{x_1} f_i + x_2\partial_{x_2} f_i = n f_i$ for $i = 1, 2$. Therefore,

$$\delta_{\mathbf{f}} W = x_1(f_1\partial_{x_1} f_2 + f_2\partial_{x_2} f_2) - x_2(f_1\partial_{x_1} f_1 + f_2\partial x_2 f_1), \tag{2.87.a}$$
$$= (\partial_{x_2} f_2 + \partial_{x_1} f_1)W. \tag{2.87.b}$$

Hence, W is a Darboux polynomial and so are its divisors. Conversely, if J is a Darboux polynomial, then $\delta_{\mathbf{f}} J = \alpha J$ and

$$(x_1\alpha - n f_1)J = -W\partial_{x_2} J, \tag{2.88.a}$$
$$(x_2\alpha - n f_2)J = W\partial_{x_1} J. \tag{2.88.b}$$

Since $W \neq 0$, we can assume that $(x_1\alpha - n f_1) \neq 0$. Since J is irreducible, J divides W. □

The result holds for the homogeneous components of the vector fields and the first integral as shown by Christopher (1994).

Lemma 2.3 (Christopher) *Let J be a Darboux polynomial of $\delta_{\mathbf{f}} = f_1\partial_{x_1} + f_2\partial_{x_2}$ and $\tilde{J}$, $\tilde{f}_1, \tilde{f}_2$ the corresponding homogeneous components of highest degree. Then, all irreducible factors of $\tilde{J}$ are factors of $x_1\tilde{f}_2 - x_2\tilde{f}_1$.*

Example 2.21 Consider the homogeneous quadratic vector field (Collins, 1995)

$$\dot{x} = x^2 + 2xy + 3y^2, \tag{2.89.a}$$

$$\dot{y} = 2y(2x + y). \tag{2.89.b}$$

The analysis of this system is straightforward. The function W of Christopher's lemma is

$$W = x(2y(2x + y)) - y(x^2 + 2xy + 3y^2) = 3y(x - y)(x + y). \tag{2.90}$$

The divisors of W provide the three Darboux polynomials

$$J_1 = (x + y), \qquad \alpha_1 = (x + 5y), \tag{2.91.a}$$

$$J_2 = (x - y), \qquad \alpha_2 = (x - y), \tag{2.91.b}$$

$$J_3 = y, \qquad \alpha_3 = 2(2x + y), \tag{2.91.c}$$

which, in turn, provide the first integral (by the relation $\alpha_1 + 3\alpha_2 - \alpha_3 = 0$)

$$I = J_1 J_2^3 J_3^{-1} = \frac{(x + y)(x - y)^3}{y}. \tag{2.92}$$

The phase portrait of system (2.89) is given in Figure 2.8. ■

2.6 Third integrals

First integrals are global invariants defined for all initial conditions. Second integrals are invariant on their zero level set. There is yet another type of invariant, the *third integrals*, which are constant on a particular level set of another integral. As an example, assume that the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in \mathbb{K}^n[\mathbf{x}]$ has a first integral $I = I(\mathbf{x})$. It may be possible to find another conserved quantity for particular values of the first integral I. For instance, at the value $I = 0$, we could look for conditions on the system for the existence of another invariant $G(x)$. That is, $G = G(\mathbf{x}) : \mathbb{K}^n \to \mathbb{K}$ and $\alpha = \alpha(\mathbf{x})$ is such that

$$\delta_{\mathbf{f}} G = \alpha I. \tag{2.93}$$

On the level set $I = 0$, the function G is an invariant. This type of *constrained* or *conditional* integrals often arises in Hamiltonian dynamics where new integrals appear when the Hamiltonian vanishes.

Example 2.22 **A three-dimensional vector field.** Consider the system

$$\dot{x} = -2x^2 + 2z, \tag{2.94.a}$$

$$\dot{y} = -3xy, \tag{2.94.b}$$

$$\dot{z} = 4xz - 2x(2x^2 - 9y^2). \tag{2.94.c}$$

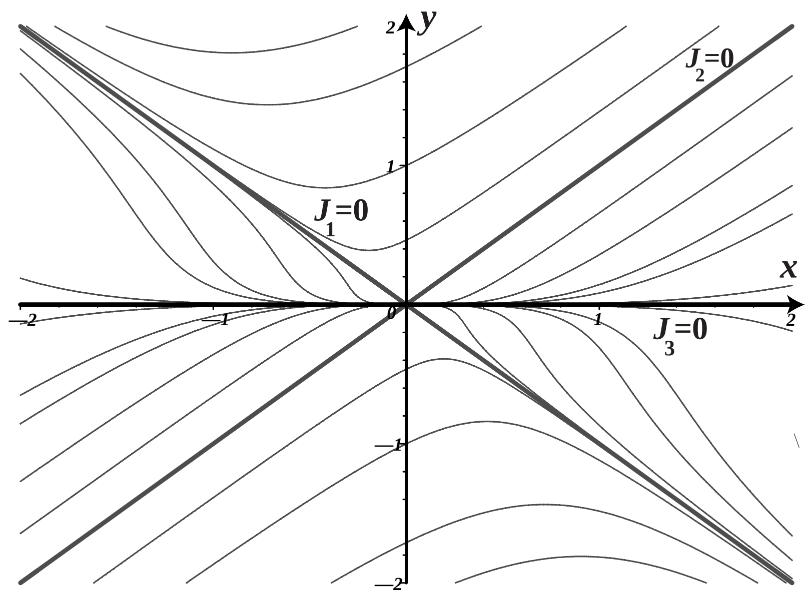

Figure 2.8: The level sets $J_i = 0$ ($i = 1, 2, 3$) of Equations (2.89) together with some level sets of $I = J_1 J_2^3 J_3^{-1}$.

A first integral can be found using the method outlined in Section 2.4.1:

$$I = z - x^2 - 3y^2. \tag{2.95}$$

This first integral can be used to foliate the phase space in a family of elliptical paraboloids centered on the z-axis. However, the flow on the paraboloid itself has to be determined. Remarkably, on the zero level set of I, we can find the third integral

$$G = z - x^2 - y^2, \tag{2.96}$$

and we verify that $\delta_f G = \alpha I$ where $\alpha = -8x$. Therefore, we conclude that the flow on $I = 0$ lies on the intersection of $I = 0$ with $G = C$ (see Figure 2.9 for $C = 4$). ■

Definition 2.15 Let $I = I(\mathbf{x})$ be a first integral of the vector field $\delta_{\mathbf{f}}$. The C^1 function $G = G(\mathbf{x})$ is a *third integral* if $\delta_{\mathbf{f}} G = \alpha I$, where $\alpha = \alpha(\mathbf{x})\colon \mathbb{K}^n \to \mathbb{K}$. It is *trivial* if $G \in \mathbb{K}$.

Example 2.23 The Goryachev-Chaplygin top. This is probably the most well-known example of a third integral. Consider once more Euler equations (2.16) for rigid body motion around a fixed point

$$J\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (J\boldsymbol{\omega}) = \mathbf{X} \times \boldsymbol{\gamma}, \tag{2.97.a}$$

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0. \tag{2.97.b}$$

It was found by Goryachev (1900) and Chaplygin (1901) that the system admits a third integral when $J_1 = J_2 = 4J_3$, the center of mass is in the plane of equal moments of inertia, ($X_2 = X_3 = 0$) and

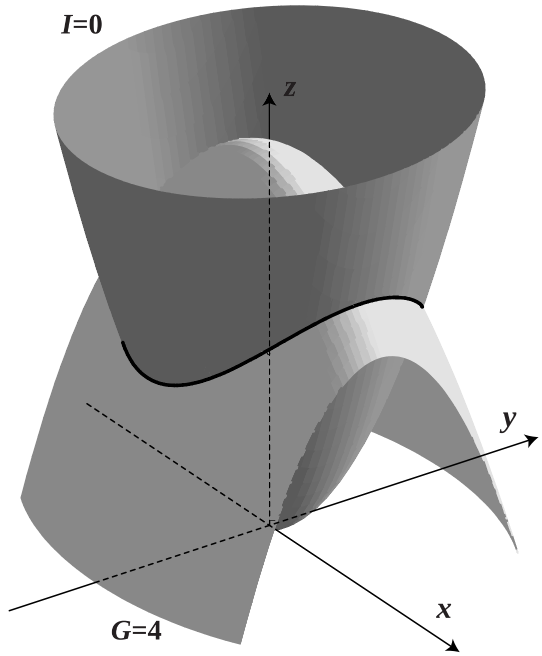

Figure 2.9: The zero level set of the first integral $I = 0$ together with the level set $G = 4$ of the third integral for Equations (2.94). The flow follows the intersection of these two paraboloids (black line). Note that the level set $G = 4$ is not an invariant set, but its intersection with $I = 0$ is an invariant set.

the first integral corresponding to the projection of the angular momentum vector on the vertical axis through zero vanishes ($I_3 = 0$ in (2.18)). The third integral is

$$G = (\omega_1^2 + \omega_2^2)\omega_1 + \frac{X_1}{J_3}\omega_1\gamma_3. \tag{2.98}$$

That is, $\delta_{\mathbf{f}} G = \alpha I_3$ with $\alpha = \frac{\omega_2 X_1}{4J_3^2}$. In this case, the equations of motion simplify considerably and a solution with four arbitrary constants can be obtained by separation of variables.

In recent years, there has been a considerable amount of work on the Goryachev-Chaplygin top in connection with integrable Yang-Mills theories (Brzezinski, 1996; Soonkeon, 1997), integrable structures in supersymmetric gauge theories (Ahn & Nam, 1996) and quantum integrable systems (Komarov & Novikov, 1994). ■

2.7 Higher integrals

First, second, and third integrals can be unified by considering them as the solution of the linear system

$$\delta_{\mathbf{f}} I_i = \sum_{j=1}^{p} \alpha_{ij} I_j, \qquad i, j = 1, \dots, p, \tag{2.99}$$

where I_i and α_{ij} are functions from $\mathbb{K}^n$ to $\mathbb{K}$. If $\alpha_{i1} = \alpha_{i2} = \ldots = \alpha_{in} = 0$, then I_i is a first integral. If $\alpha_{ij} \in \mathbb{K}$ and $\alpha = (\alpha_{ij})$ can be diagonalized, then $I_i e^{\mu_i t}$ are time-dependent first integrals. In general, if $\alpha_{ij} \notin \mathbb{K}$, then this system can be used to define second, third or higher integrals.

Example 2.24 The Goryachev-Chaplygin top revisited. Consider again the Euler-Poisson equation (2.97) in the Goryachev-Chaplygin case $J_1 = J_2 = 4J_3$, $X_2 = X_3 = 0$. Let I_1, I_2, I_3 be the three classical first integrals (2.18), and $I_4 = G$ the third integral found by Goryachev (Equation (2.98)). Further let,

$$I_5 = \frac{X_1^2 \omega_1^2}{(J_3^2 \omega_1^2 + \omega_2^2)^3}. \tag{2.100}$$

This function was shown by Goryachev to vanish identically whenever $G = 0$. That is, we can recast these five integrals in the form (2.99) with

$$\dot{I}_1 = 0, \tag{2.101.a}$$
$$\dot{I}_2 = 0, \tag{2.101.b}$$
$$\dot{I}_3 = 0, \tag{2.101.c}$$
$$\dot{I}_4 = \alpha_{43} I_3, \tag{2.101.d}$$
$$\dot{I}_5 = \alpha_{54} I_4, \tag{2.101.e}$$

where

$$\alpha_{43} = \frac{\omega_2 X_1}{4J_3^2}, \tag{2.102.a}$$
$$\alpha_{54} = \frac{3X_1^2 \omega_2 \omega_1}{J_3^2 (\omega_1^2 + \omega_2^2)^4}. \tag{2.102.b}$$

Note that I_5 is not a third integral since it vanishes only when $I_4 = 0$. ■

2.8 Class-reduction

One of the most interesting aspects concerning the existence of first integrals lies in the possibility of *class-reduction*, namely that the existence of a first integral in one class of functions implies the existence of a first integral in a sub-class of functions. Probably the most well-known result on class-reduction is Brun's theorem (1887) (see for instance Falk (1900) or Kummer *et al.* (1990) for a modern treatment). It shows that the existence of an algebraic first integral implies the existence of a rational first integral. In addition, we provide three different reduction theorems.

Theorem 2.4 (Brun) *If the vector field $\delta_{\mathbf{f}}$ has k non-trivial independent algebraic first integrals, then it has k non-trivial independent rational first integrals.*

Proof. Let $I = C$ be an algebraic first integral and let

$$P(I) = q_0 + q_1 C + \ldots + q_{s-1} C^{s-1} + C^s \tag{2.103}$$

be its minimal polynomial, where $q_i \in \mathbb{K}(\mathbf{x})$. That is, $I = I(\mathbf{x})$ is defined as the roots of $P(C) = 0$. Since I is non trivial, one of the rational function q_i is not constant. Let i be such that $q_i \in \mathbb{K}(\mathbf{x}) \setminus \mathbb{K}$. Since $\delta_{\mathbf{f}}(C) = 0$, the derivative of $P(I)$ along the flow is

$$\delta_{\mathbf{f}} P(I) = \delta_{\mathbf{f}} q_0 + C \delta_{\mathbf{f}} q_1 + \ldots + C^{s-1} \delta_{\mathbf{f}} q_{s-1} = 0. \tag{2.104}$$

But, since P is minimal, we have $\delta_{\mathbf{f}} q_0 = \delta_{\mathbf{f}} q_1 = \ldots = \delta_{\mathbf{f}} q_s = 0$. Therefore, $\delta_{\mathbf{f}} q_i = 0$ and q_i is a first integral. Now, let $I' = C'$ be another independent first integral whose minimal polynomial is

$$P(C') = q'_0 + q'_1 C' + \ldots + q'_{s'-1} C'^{s'-1} + C'^{s'}. \tag{2.105}$$

Again each q'_j is a first integral. The independence between the two first integrals implies that there exist $i < s$ and $i' < s'$ such that q_i, $q'_i \in \mathbb{K}(\mathbf{x}) \setminus \mathbb{K}$ are two non-trivial independent first integrals. By induction, k independent rational first integrals can be built. $\square$

In the case of a weight-homogeneous vector field, every weight-homogeneous component of a first integral is itself a first integral. This is a direct consequence of Proposition 2.3.

Proposition 2.7 *Let I be a first integral of $\delta_{\mathbf{f}}$, a weight-homogeneous vector field of weight $\mathbf{w}$. Then, every weight-homogeneous component of I is a first integral of $\delta_{\mathbf{f}}$.*

The quasimonomial integrals, that is, first integrals that are the sum of quasimonomials, $I = \sum_i a_i \mathbf{x}^{\boldsymbol{\alpha}_i}$, can be simplified when they arise as first integrals of an analytic vector field.

Proposition 2.8 *Let $I = \sum_i a_i \mathbf{x}^{\boldsymbol{\alpha}_i}$ be a quasimonomial first integral of $\delta_{\mathbf{f}}$, where $\delta_{\mathbf{f}}$ is an analytic vector field. Then, there exists a first integral of the form*

$$\widehat{I} = \mathbf{x}^{\boldsymbol{\beta}} P(\mathbf{x}), \tag{2.106}$$

where $P = P(\mathbf{x})$ is polynomial.

Proof. Compute the time derivative of I:

$$\begin{aligned} \dot{I} &= \sum_{i=1}^{m} \mathbf{x}^{\boldsymbol{\alpha}_i} \left(\sum_{j=1}^{n} \frac{\alpha_{ij} f_j}{x_j} \right), \\ &= \sum_{i=1}^{m} h_i(\mathbf{x}). \end{aligned} \tag{2.107}$$

Consider the first term in the sum. Two things can happen. If $\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_k \notin \mathbb{Z}^n$, none of the quasi-monomial terms in h_1 and h_k match and there is no possibility of cancellation. The only possibility of cancellation arises if $\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_k \in \mathbb{Z}^n$. In this case we can write $h_1 + h_k = \mathbf{x}^{\boldsymbol{\beta}} h(\mathbf{x})$ where $h(\mathbf{x}) \in \mathbb{K}^n(\mathbf{x})$. Now, consider all the $\boldsymbol{\alpha}'_k s$ that satisfy a commensurate relation with $\boldsymbol{\alpha}_1$ and let $\mathcal{K} = \{k | \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_k \in \mathbb{Z}^n\}$ and $\boldsymbol{\beta}$ such that

$$\boldsymbol{\alpha}_k = \boldsymbol{\beta} + \mathbf{m}_k, \qquad \mathbf{m}_k \in \mathbb{N}^n, k \in \mathcal{K}. \tag{2.108}$$

Since I is a first integral, $\dot{I} = 0$ and

$$\widehat{I} = \mathbf{x}^{\boldsymbol{\beta}} \sum_{k \in \mathcal{K}} a_k \mathbf{x}^{\mathbf{m}_k} \tag{2.109}$$

is also first integral. □

The most important result for class-reduction is the Prelle-Singer theorem (Prelle & Singer, 1983). This beautiful theorem states that the existence of an elementary first integral for a polynomial vector field implies the existence of a logarithmic first integral. The general form of the theorem uses the terminology of Section 2.2.2.

Theorem 2.5 *Let $\{k, \Delta\}$ be a differential field of characteristic zero and let $\{K, \Delta'\}$ be an elementary extension of $\{k, \Delta\}$ with the same sets of constants. Let $D = \delta_{\mathbf{f}} = \mathbf{f}.\partial_x$ for $\partial_{x_i} \in k$ and $f_i \in K$. Assume that $C(K, \Delta')$ is strictly included in $C(K, D)$. Then, there exist elements $g_0, g_1, \dots, g_m$ of K algebraic over k and $c_1, \dots, c_m$ in $C(k, \Delta)$ such that*

$$Dg_0 + \sum_{i=1}^{m} c_i \frac{Dg_i}{g_i} = 0, \tag{2.110}$$

and

$$\delta g_0 + \sum_{i=1}^{m} c_i \frac{\delta g_i}{g_i} \neq 0 \quad \textit{for some } \delta \in \Delta. \tag{2.111}$$

Therefore, we conclude that if a vector field $\delta_{\mathbf{f}}$ has an elementary first integral $I(\mathbf{x}) = C$ (that is, roughly speaking, a function $I(\mathbf{x})$ obtained by finitely many composition of algebraic, exponential and logarithmic operations), then the vector field also has a logarithmic first integral (that is, a function $\tilde{I}(\mathbf{x})$ obtained as a finite sum of logarithms of algebraic functions). This fundamental theorem restricts the class of possible first integrals obtained by elementary manipulations to the class of logarithmic first integrals.

The construction of Darboux polynomials given in Section 2.5.1 gives an explicit but not decidable way of building logarithmic first integrals for planar vector fields specified in the Prelle-Singer theorem. It was also shown by Prelle and Singer that for planar vector fields the existence of an elementary first integral implies that the system admits a Darboux polynomial and that the integrating factor can be explicitly computed (see Proposition 2.6).

The analysis of Prelle and Singer has been extended to Liouvillian first integrals (Singer, 1992). If the general solution of a vector field satisfies a Liouvillian relation (that is, there is a Liouvillian function of several variables vanishing on solution curves), then the system admits a Liouvillian first integral on a nonempty open set. Moreover, for planar vector fields, the form of such a Liouvillian first integral can be specified.

2.9 First integrals for vector fields in $\mathbb{R}^3$: the compatibility analysis

The compatibility analysis is based on Frobenius' integrability theorem (Choquet-Bruhat, 1968, p.192). It allows us to find conditions on the parameters of a vector field for the existence of time-independent first integrals and provides an explicit method for their computation. The calculations involved are quite tedious and it is only following the development of computer algebra that the method has been successfully implemented (Strelcyn & Wojciechowski, 1988; Grammaticos *et al.*, 1990a; Moulin-Ollagnier, 1990).

Consider two differentiable vector fields $\delta_{\mathbf{f}} = \mathbf{f}.\partial_{\mathbf{x}}$ and $\delta_{\mathbf{g}} = \mathbf{g}.\partial_{\mathbf{x}}$ in $\mathbb{R}^3$ together with their corresponding systems of ODEs

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \text{and} \quad \dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}). \tag{2.112}$$

The *Lie bracket* $[.,.]$ between two such vector fields is defined by the vector field whose components are

$$[\mathbf{f},\mathbf{g}]_i = (\delta_{\mathbf{f}}\mathbf{g} - \delta_{\mathbf{g}}\mathbf{f})_i = \sum_{j=1}^{3}(\frac{\partial g_i}{\partial x_j}f_j - \frac{\partial f_i}{\partial x_j}g_j). \tag{2.113}$$

Theorem 2.6 (Frobenius) *Assume that $\delta_{\mathbf{f}}$ and $\delta_{\mathbf{g}}$ admit the same non-trivial globally defined time-independent first integral $I = I(\mathbf{x})$, that is, $\mathbf{f}.\partial_{\mathbf{x}}I = \mathbf{g}.\partial_{\mathbf{x}}I = 0$. Then, the three vector fields $\mathbf{f},\mathbf{g}$ and $[\mathbf{f},\mathbf{g}]$ are linearly dependent for all points $\mathbf{x} \in \mathbb{R}^3$.*

Frobenius' theorem implies the existence of $\alpha = \alpha(\mathbf{x})$, $\beta = \beta(\mathbf{x})$ and $\gamma = \gamma(\mathbf{x})$ such that

$$\alpha\mathbf{f} + \beta\mathbf{g} + \gamma[\mathbf{f},\mathbf{g}] = 0 \qquad \forall\ \mathbf{x} \in \mathbb{R}^3. \tag{2.114}$$

Or, equivalently $\det\{\mathbf{f},\mathbf{g},[\mathbf{f},\mathbf{g}]\} = 0$.

Definition 2.16 Two vector fields $\delta_{\mathbf{f}}$, $\delta_{\mathbf{g}}$ in $\mathbb{R}^3$ are said to be *compatible* if

$$\det\{\mathbf{f},\mathbf{g},[\mathbf{f},\mathbf{g}]\} = 0. \tag{2.115}$$

Conversely, the compatibility condition can be used as an integrability condition. If the compatibility condition is satisfied, the vector fields $\mathbf{f}$ and $\mathbf{g}$ are parallel (at least locally). Hence, there exists a local first integral that both systems admit. Now, we can look for a function I, globally defined, such that the compatibility condition is satisfied. The global existence of such a function is, however, not guaranteed and this method, like the previous ones, only provides information on the system when successfully applied. Condition (2.114) with $\beta = 0$, (*i.e.*, $[\mathbf{f},\mathbf{g}] = \alpha\mathbf{f}$), is exactly the *Lie invariance condition* generated by the vector field $\delta_{\mathbf{g}}$. That is, if this condition is satisfied, then all solutions of $\delta_{\mathbf{f}}$ are invariant under the symmetry generated by $\delta_{\mathbf{g}}$ (Olver, 1993).

Therefore, for a given $\delta_{\mathbf{f}}$ we look for a vector field $\delta_{\mathbf{g}}$ with known first integrals for which the compatibility condition with $\delta_{\mathbf{f}}$ is satisfied. The simplest choice for $\delta_{\mathbf{g}}$ is a linear vector field. Assume that $u = u(\mathbf{x})$ and $v = v(\mathbf{x})$ are two independent first integrals of $\delta_{\mathbf{g}}$, that is, $\mathbf{g}.\partial_{\mathbf{x}}u = \mathbf{g}.\partial_{\mathbf{x}}v = 0$. Then, any first integral of $\mathbf{g}$ is functionally dependent on u and v. In particular, the common first integral between $\delta_{\mathbf{g}}$ and $\delta_{\mathbf{f}}$ can be written $I = I(u,v)$ and satisfies

$$\begin{aligned} \mathbf{f}.\partial_{\mathbf{x}}I &= (\partial_u I)\mathbf{f}.\partial_{\mathbf{x}}u + (\partial_v I)\mathbf{f}.\partial_{\mathbf{x}}v = 0, \\ &= \mathbf{f}.\partial_{\mathbf{x}}v\,((\partial_u I)G(u,v) + \partial_v I)\,, \end{aligned} \tag{2.116}$$

where $G(u,v) = \frac{\mathbf{f}.\partial_{\mathbf{x}}u}{\mathbf{f}.\partial_{\mathbf{x}}v}$. The fact that $G(u,v)$ is a function of u,v only follows from the compatibility condition and the fact that $G(u,v)$ itself is a first integral of $\mathbf{f}$. Now, in order to solve relation (2.116) we have to solve

$$\frac{du}{dv} = G(u,v). \tag{2.117}$$

The first integral $I = I(u,v)$ of this last equation is the common integral of $\delta_{\mathbf{f}}$ and $\delta_{\mathbf{g}}$ given by $I = I(u(\mathbf{x}),v(\mathbf{x}))$. Therefore, the algorithm to find new integrals is

1. Find a compatible (linear) vector field $\delta_{\mathbf{g}}$.
2. Find two first integrals u, v of $\delta_{\mathbf{g}}$.
3. Compute $G(u,v) = (\mathbf{f}.\partial_{\mathbf{x}}u)/(\mathbf{f}.\partial_{\mathbf{x}}v)$
4. Find a first integral $I = I(u,v)$ of the first order equation $\frac{du}{dv} = G(u,v)$.

In order to illustrate the techniques and difficulties associated with this method we apply it on a simple example.

Example 2.25 The Lotka-Volterra ABC system. Consider again Equations (2.23)

$$\dot{x}_1 = x_1(Cx_2 + x_3), \tag{2.118.a}$$
$$\dot{x}_2 = x_2(x_1 + Ax_3), \tag{2.118.b}$$
$$\dot{x}_3 = x_3(Bx_1 + x_2), \tag{2.118.c}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and A, B, C are parameters. The simplest possible ansatz for the vector field $\delta_{\mathbf{g}}$ is (Grammaticos *et al.*, 1990a)

$$\mathbf{g}(\mathbf{x}) = L\mathbf{x}, \tag{2.119}$$

where L is a 3×3 matrix with constant coefficients. The compatibility condition between $\delta_{\mathbf{f}}$ and $\delta_{\mathbf{g}}$ reads

$$\det \begin{bmatrix} x_1(Cx_2+x_3) & L_{11}x_1+L_{12}x_2+L_{13}x_3 & P(x_1,x_2,x_3) \\ x_2(x_1+Ax_3) & L_{21}x_1+L_{22}x_2+L_{23}x_3 & Q(x_1,x_2,x_3) \\ x_3(Bx_1+x_2) & L_{31}x_1+L_{32}x_2+L_{33}x_3 & R(x_1,x_2,x_3) \end{bmatrix} = 0, \ \forall\, \mathbf{x} \in \mathbb{R}^3. \tag{2.120}$$

In the last expression, P, Q, R are the quadratic homogeneous polynomials defined by

$$[\mathbf{f}, \mathbf{g}] = (P, Q, R). \tag{2.121}$$

The compatibility condition (2.120) is a polynomial of fifth degree in $\mathbf{x}$ and fourth degree in the unknowns L_{ij}, A, B, C. Eventually, a set of 18 non trivial quadratic equations in the entries L_{ij} can be derived and discussed in terms of the parameters A, B, C. It clearly appears in this simple example that it would be difficult to perform this cumbersome analysis without the use of computer algebra. As an example, if we take $A =$ and $C \neq 0$, the matrix $L = \mathrm{diag}(a, a, 0)$ (a an arbitrary parameter) defines a compatible vector field with first integrals $u = x_1/x_2$ and $v = x_3$. The corresponding first integral of the Lotka-Volterra system is then

$$I = |x_3|^C \left|\frac{x_2}{x_1}\right| \left|C - \frac{x_1}{x_2}\right|^{BC+1}. \tag{2.122}$$

■

The advantage of the compatibility condition method is that it is constructive and that the type of first integrals obtained is not constrained by an initial ansatz. Basically, the first integrals found are those of the vector field $\delta_{\mathbf{g}}$. Hence, transcendental first integrals can be obtained (this procedure was used to find some of the first integrals of Example 2.4). However, the method is also limited in many regards. Essentially, the algebra is tractable only for quadratic vector fields in $\delta_{\mathbf{f}}$ and linear in $\delta_{\mathbf{g}}$. This method was developed further by Moulin-Ollagnier (1996) for homogeneous vector fields (the so-called *extended compatibility method*).

2.10 Integrability

In this section, we discuss different notions related to integrability. Integrability is generally understood as the existence of sufficiently many first integrals to render the integration of the differential equations possible. This rather vague notion will now be defined.

2.10.1 Local integrability

The question of local integrability is trivial in the sense that there always exist first integrals locally. Local integrability is guaranteed by the following two results.

Proposition 2.9 *Let $\mathbf{A} = (I_1, \ldots, I_n)$ be n time-dependent first integrals of $\delta_{\mathbf{f}}$ of class C^1 on an open subset $V \subset (U \times \mathbb{R})$ where the Jacobian matrix $D\mathbf{A} = (\partial_{\mathbf{x}} I_1, \partial_{\mathbf{x}} I_2, \ldots, \partial_{\mathbf{x}} I_n)$ is invertible. Let $(\mathbf{x}_0, t_0) \in V$ and $J_i = I_i(\mathbf{x}_0, t)$. Now consider $\mathbf{x} = \mathbf{x}(t, \mathbf{A})$ the inverse function of $\mathbf{A}$ for $(\mathbf{x}, t)$ near $(\mathbf{x}_0, t_0)$. Then, $\mathbf{x} = \mathbf{x}(t, \mathbf{A})$ is a solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ for any constant value of $\mathbf{A}$. Moreover, a C^1 function $u = u(\mathbf{x}, t)$ defined on V near $(\mathbf{x}_0, t_0)$ is a first integral of $\delta_{\mathbf{f}}$ if and only if there exists a C^1 function $v = v(\mathbf{A})$ such that $u(\mathbf{x}, t) = v(\mathbf{A}(\mathbf{x}, t))$ in a neighborhood of $(\mathbf{x}_0, t_0)$.*

Proof. Since $A_i = I_i$ is a first integral for all i, we have

$$\frac{\partial \mathbf{A}}{\partial t} + (D\mathbf{A})\mathbf{f} = 0, \tag{2.123}$$

Since the matrix $D\mathbf{A}$ is invertible locally, we have $(D\mathbf{A})^{-1}\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{f}$. Now, if $\mathbf{x}$ is the inverse function of $\mathbf{A}$, we have $\dot{\mathbf{x}} = -(D\mathbf{A})^{-1}\frac{\partial \mathbf{A}}{\partial t} = \mathbf{f}$. Hence, $\mathbf{x} = \mathbf{x}(t, \mathbf{A})$ is a solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ for all constant first integrals $\mathbf{A}$.

If $v = v(\mathbf{A})$ is a C^1 function of $\mathbf{A}$, then v is a first integral of $\delta_{\mathbf{f}}$. Conversely, assume that $u = u(\mathbf{x}, t)$ is a first integral. It can be written as $u = u(\mathbf{x}(\mathbf{A}, t), t) = v(\mathbf{A}, t)$ in a neighborhood of $(\mathbf{x}_0, t_0)$. We have to prove that $\partial_t v = 0$. By definition, $\partial_t v = \partial_t u + \mathbf{x}.\partial_{\mathbf{x}} u$, but since u is a first integral and $\mathbf{x}$ is a solution, $\partial_t v = 0$. □

This proposition shows that the local solution can be found from n independent first integrals. Conversely, if the solution exists and is unique, we can define n independent first integrals locally.

Proposition 2.10 *Let $\mathbf{f}$ be C^0 on an open subset $V \subset (U \times \mathbb{R})$. If the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ has a unique C^1 solution, then the vector field $\delta_{\mathbf{f}}$ has n independent first integrals of class C^1 in a neighborhood of a point $(\mathbf{x}_0, t_0) \in V$.*

Proof. Let $\mathbf{x} = \boldsymbol{\varphi}(t, t_0, \mathbf{x}_0)$ be the local solution of the initial value problem and consider $\mathbf{y} = \boldsymbol{\varphi}(t, t_0, \mathbf{x})$. Then, y_i is a first integral of $\delta_{\mathbf{f}}$ in a neighborhood of $(\mathbf{x}_0, t_0)$ since it assumes the constant value $y_i = x_i(t_0)$. Moreover $D\mathbf{y}(t_0)$ is the identity matrix. Hence, $D\mathbf{y}(t)$ is invertible in a neighborhood of t_0. □

Therefore, we conclude that the problem of finding local first integrals is trivial in the sense that the initial value problem always provides local first integrals that can be built from the unique local solution. Therefore, the problem of integrability is generally understood as the problem of finding define global first integrals.

2.10.2 Liouville integrability

In this section, we give a concise overview of the simplest case of Liouville integrability for Hamiltonian systems. A thorough discussion of Hamiltonian systems and Arnold-Liouville integrability is given in Chapter 6. Consider an n-degree-of-freedom Hamiltonian $H = H(\mathbf{p}, \mathbf{q})$ which is analytic in $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$. The Hamilton equations read

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad i = 1, \dots, n, \tag{2.124.a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \dots, n. \tag{2.124.b}$$

Definition 2.17 The Hamiltonian $H(\mathbf{p}, \mathbf{q}) = 0$ is *Liouville integrable* if there exist n independent analytic first integrals $I_1 = H, I_2, \dots, I_n$ in involution (*i.e.*, $\{I_i, I_j\} = 0$).[8]

Moreover, if the manifolds defined by the intersection of their level sets

$$\bigcap_{i=1}^{n} \{I_i = a_i,\ (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^{2n}\}, \tag{2.125}$$

are compact and connected, then Theorem 6.3 due to Arnold (1988b) states that they are topologically real tori.

2.10.3 Algebraic integrability

Different notions of complete integrability are used in the literature. The weaker definition is an extension of Arnold-Liouville integrability to general vector fields.

Definition 2.18 A vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in \mathbb{K}^n$ is *algebraically integrable in the weak sense* if there exist k independent algebraic first integrals $I_i(\mathbf{x}) = K_i$ with $i = 1, \dots, k$. These k first integrals define an $(n-k)$-dimensional algebraic variety. In addition, there must exist $(n-1-k)$ other independent first integrals given by the integral of a total differential defined on the algebraic variety,

$$J_i(\mathbf{x}) = \sum_{j=1}^{n-k} \int^{x_j} \varphi_{ij}(\mathbf{x}) dx_j, \qquad i = 1, \dots, n-1-k, \tag{2.126}$$

where $\varphi_{ij}(\mathbf{x})$ are algebraic functions of $\mathbf{x}$.

This definition seems to be useless for systems which are not Hamiltonian since the existence of total differentials is not known a priori. However, as shown in the next section, such a situation may occur. A first example was provided by the Euler equations where only four first integrals are required to complete the integration, the fifth one being given by the integral of a total differential. However, it is understood that this definition is, in most cases, likely to be applicable to Hamiltonian systems where the symplectic structure provides through Arnold-Liouville's theorem an explicit way

[8] The brackets { , } are the usual *Poisson Brackets* $\{I, J\} = \sum_i \left(\frac{\partial I}{\partial p_i} \frac{\partial J}{\partial q_i} - \frac{\partial I}{\partial q_i} \frac{\partial J}{\partial p_i} \right)$.

of constructing the extra first integrals. The stronger definition of algebraic integrability is equivalent to the weak definition with the condition $k = n - 1$.

Definition 2.19 A vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in \mathbb{K}^n$ is *algebraically integrable* if there exist $(n-1)$ independent algebraic first integrals $I_i,\ i = 1, \ldots, n$.

Still, there exist other definitions of algebraic integrability for Hamiltonian systems (see Chapter 6), the so-called *algebraic complete integrability* introduced by Adler and van Moerbeke (1987; 1989b) and the *hyperelliptically separable systems* by Ercolani and Siggia (1989; 1989; 1991). Their definitions cover systems which can be integrated in terms of Abelian functions (see Section 6.3).

2.11 Jacobi's last multiplier method

The last multiplier method of Jacobi gives a constructive way to build a first integral for an n-dimensional system with a local conserved density when $(n-2)$ first integrals are known. The method is best illustrated on a planar vector field

$$\dot{x} = f(x, y), \tag{2.127.a}$$
$$\dot{y} = g(x, y). \tag{2.127.b}$$

One way to solve this system is to find a *Jacobi multiplier*, a C^1 (non-zero) function $M = M(x, y)$, such that

$$M(gdx - fdy) = dI. \tag{2.128}$$

Since $g(x, y)dx = f(x, y)dy$ on all solutions, we have $dI = 0$ and hence $I = I(x, y)$ is a first integral. This first integral can be found by equating the last relation with $dI = \partial_x I dx + \partial_y I dy$. That is,

$$\frac{\partial I}{\partial x} = Mg, \tag{2.129.a}$$
$$\frac{\partial I}{\partial y} = -Mf. \tag{2.129.b}$$

Hence, the first integral reads

$$I(x, y) = -\int^y Mf\ dx + J(x) \tag{2.130.a}$$
$$J'(x) = Mg + \frac{\partial}{\partial x}\int^y Mf\ dy. \tag{2.130.b}$$

The relation (2.129), together with $\partial^2_{xy} I = \partial^2_{yx} I$ also implies that

$$\partial_x(Mf) + \partial_y(Mg) = 0. \tag{2.131}$$

The function M is also called the *density of the integral invariant* (Andronov *et al.*, 1971, p. 402) since on any bounded closed region D in the phase plane $\mathbb{R}^2$ we have

$$\iint_D M(x, y)dxdy = \iint_{\varphi_t(D)} M(x, y)dxdy, \tag{2.132}$$

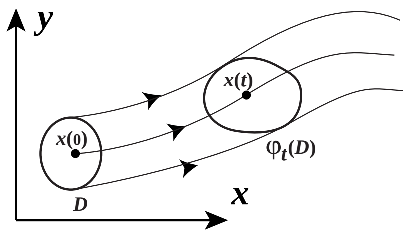

Figure 2.10: Evolution of a bounded closed region D under the flow.

where $\varphi_t(D)$ is the transformation of the domain D under the flow generated by the solution (see Figure 2.10). Therefore, to find a first integral $I = I(x, y)$, we look for a multiplier M, given by any non-trivial solution of $\partial_x(Mf) + \partial_y(Mg) = 0$.

A similar result holds in n dimensions. If a Jacobi multiplier is known together with $(n-2)$ first integrals, we can reduce locally the $(n-2)$-dimensional system to a two-dimensional vector field on the intersection of the $(n-2)$ level sets formed by the first integrals. The previous result then implies the existence of an extra first integral. Consider an n-dimensional vector field $\delta_{\mathbf{f}} = \mathbf{f}.\partial_{\mathbf{x}}$ and the corresponding system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^n$.

Definition 2.20 Let $M = M(\mathbf{x})$ be a non-negative C^1 function non-identically vanishing on any open subset of $\mathbb{R}^n$, then M is a *Jacobi multiplier* of $\delta_{\mathbf{f}}$ if

$$\int_D M(\mathbf{x})d\mathbf{x} = \int_{\varphi_t(D)} M(\mathbf{x})d\mathbf{x}, \tag{2.133}$$

where D is any open subset of $\mathbb{R}^n$ and $\varphi_t(.)$ is the *flow* generated by the solution $\mathbf{x} = \mathbf{x}(t)$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

The multiplier M can also be seen as the density associated with the invariant measure $\int_D M d\mathbf{x}$. Given a function M, it is not trivial to verify (2.133) since the solution is required. However, we can use the generalization of (2.131) (Whittaker, 1944).

Proposition 2.11 *Let $M = M(\mathbf{x})$ be a non-negative C^1 function non-identically vanishing on some open subset of $\mathbb{R}^n$. Then, $M = M(\mathbf{x})$ is a Jacobi multiplier if and only if*

$$\partial_{\mathbf{x}}(M\mathbf{f}) = 0. \tag{2.134}$$

In order to build a first integral from the Jacobi multiplier, we need the following lemma.

Lemma 2.4 *Let $\mathbf{y} = \mathbf{y}(\mathbf{x})$ be an invertible change of variables mapping the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ to $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$, then*

$$\partial_{\mathbf{x}}\mathbf{f} = \frac{1}{\gamma}\partial_{\mathbf{y}}(\gamma\mathbf{g}), \tag{2.135}$$

where γ is the determinant of the Jacobian matrix $J = \partial_{\mathbf{y}}\mathbf{x}$.

Proof. The left hand side of (2.135) gives

$$\begin{aligned}
\partial_{\mathbf{x}}\mathbf{f} &= \partial_{\mathbf{x}}(\partial_{\mathbf{y}}\mathbf{x})\mathbf{g}, \\
&= (\partial_{\mathbf{x}}\mathbf{y})\partial_{\mathbf{y}}\left[(\partial_{\mathbf{y}}\mathbf{x})\mathbf{g}\right], \\
&= J^{-1}\left[\partial_{\mathbf{y}}(J\mathbf{g})\right], \\
&= J^{-1}\left[(\partial_{\mathbf{y}}J).\mathbf{g} + J(\partial_{\mathbf{y}}\mathbf{g})\right], \\
&= J^{-1}\partial_{\mathbf{y}}(J).\mathbf{g} + \partial_{\mathbf{y}}\mathbf{g}, \\
&= \frac{1}{\gamma}(\partial_{\mathbf{y}}\gamma)\mathbf{g} + \partial_{\mathbf{y}}\mathbf{g}, \\
&= \frac{1}{\gamma}\partial_{\mathbf{y}}(\gamma\mathbf{g}),
\end{aligned} \tag{2.136}$$

where we have used the standard identity $\sum_{kl} J^{-1}_{ik}\partial_{y_l}(J_{kj})g_l = \frac{1}{\gamma}(\partial_{y_i}(\gamma))g_j$. □

Now, assume that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ admits $(n-2)$ first integrals, $I_i(\mathbf{x}) = c_i$ with $i = 1, \ldots, n-2$. These first integrals define, up to a relabeling of the variables, an invertible transformation mapping $(x_1,\ldots,x_n)$ to $(c_1,\ldots,c_{n-2},x_{n-1},x_n)$ given by

$$y_i = I_i(\mathbf{x}), \qquad i = 1, \ldots, n-2, \tag{2.137.a}$$

$$y_{n-1} = x_{n-1}, \tag{2.137.b}$$

$$y_n = x_n. \tag{2.137.c}$$

Let Δ be the Jacobian of the transformation

$$\Delta = \begin{vmatrix} \partial_{x_1}I_1 & \partial_{x_2}I_1 & \cdots & \partial_{x_{n-2}}I_1 \\ \partial_{x_1}I_2 & \partial_{x_2}I_2 & \cdots & \partial_{x_{n-2}}I_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1}I_{n-2} & \partial_{x_2}I_{n-2} & \cdots & \partial_{x_{n-2}}I_{n-2} \end{vmatrix}. \tag{2.138}$$

The invertibility of the mapping can be used to reduce the n-dimensional flow to a two-dimensional flow on the level sets of the first integrals. By assumption, the reduced flow admits a reduced Jacobi multiplier, which, in turn, provides the last first integral.

Theorem 2.7 (Jacobi) *Consider an n-dimensional vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and assume that it admits a Jacobi multiplier $M = M(\mathbf{x})$ and $(n-2)$ first integrals $I_i(\mathbf{x}) = c_i$, $i = 1, \ldots, n-2$. Then, the system admits an extra first integral given by*

$$I_{n-1} = \int \frac{\tilde{M}}{\tilde{\Delta}}\left(\tilde{f}_n dx_{n-1} - \tilde{f}_{n-1}dx_n\right), \tag{2.139}$$

where $\tilde{(\ )}$ denotes quantities expressed in the variables $(c_1,\ldots,c_{n-2},x_{n-1},x_n)$.

Proof. Consider the change of variables (2.137). By Lemma 2.4, we have

$$\begin{aligned}
\partial_{\mathbf{x}}\mathbf{f} &= \Delta\partial_{\mathbf{y}}\left(\frac{\mathbf{f}}{\tilde{\Delta}}\right), \\
&= \Delta\left(\partial_{x_{n-1}}\frac{\tilde{f}_{n-1}}{\tilde{\Delta}} + \partial_{x_n}\frac{\tilde{f}_n}{\tilde{\Delta}}\right).
\end{aligned} \tag{2.140}$$

From Proposition 2.11, we also have

$$\begin{aligned}\partial_{\mathbf{x}}\mathbf{f} &= -\frac{1}{M}\,\mathbf{f}.\partial_{\mathbf{x}}M, \\ &= -\frac{1}{M}\left((\partial_{x_{n-1}}\tilde{M})\tilde{f}_{n-1} + (\partial_{x_n}\tilde{M})f_n\right). \qquad (2.141)\end{aligned}$$

Comparing the last two equations, we have

$$\partial_{x_{n-1}}\left(\frac{\tilde{f}_{n-1}\tilde{M}}{\tilde{\Delta}}\right) + \partial_{x_n}\left(\frac{\tilde{f}_n\tilde{M}}{\tilde{\Delta}}\right) = 0. \qquad (2.142)$$

We conclude that $(\tilde{f}_{n-1}dx_n - \tilde{f}_n dx_{n-1})\tilde{M}/\tilde{\Delta}$ is a total differential and the desired result follows. □

Example 2.26 The Lotka-Volterra ABC system. Consider the Lotka-Volterra system (Grammaticos *et al.*, 1990a) studied in Section 2.2

$$\dot{x} = x(Cy + z), \qquad (2.143.a)$$
$$\dot{y} = y(x + Az), \qquad (2.143.b)$$
$$\dot{z} = z(Bx + y). \qquad (2.143.c)$$

We can easily verify that $M = \frac{1}{xyz}$ is a Jacobi multiplier for all values of A, B, and C. Therefore, only one integral is needed to complete the integration. For instance, if $ABC + 1 = 0$, we have

$$I_1 = ABx + y - Az. \qquad (2.144)$$

Now let $x_1 = y, x_2 = z, x_3 = x$, then $\Delta = \tilde{\Delta} = \partial_y I_1 = 1$ and $\tilde{M} = (zx(I_1 - ABx + Az))^{-1}$. The extra integral is

$$I_2 = \int \frac{C(I_1 - ABx + Az) + z}{z(I_1 - ABx + Az)}dz - \int \frac{Bx + (I_1 - ABx + Az)}{x(I_1 - ABx + Az)}dx, \qquad (2.145)$$

which, after integration, leads to

$$I_2 = C\log|z| - \log|x| + \frac{1}{A}\log|y|. \qquad (2.146)$$

Note that starting from I_2 we could recover I_1 through the same process. This example is quite simple due to the linearity of the first integral. In general, the inversion of $I_1(x, y, z) = c_1$ in terms of one of the variables is much more involved and the resulting extra first integral is given by the integral of an algebraic function. ■

Example 2.27 The Euler equations. The equations (2.16) for the motion of a rigid body admit a Jacobi multiplier. We can verify that the equations are *conservative*. That is,

$$\partial_{\boldsymbol{\omega}}\dot{\boldsymbol{\omega}} + \partial_{\boldsymbol{\gamma}}\dot{\boldsymbol{\gamma}} = 0. \qquad (2.147)$$

Hence, we can take $M = 1$ as a Jacobi multiplier, the flow associated with the solution is called *divergence-free*. That is, by Liouville's theorem on the evolution of phase space volume, the flow preserves the phase space volume. In the case of Euler's equations, the equations of motion are six-dimensional and there exist three first integrals for all values of the parameters. Hence, to complete the integration and obtain a completely integrable system a fourth first integral is needed. This happens in the four classical cases given in Example 2.3. In these four cases, the fifth first integral can then be found by using Jacobi's last multiplier formula. However, this last first integral is usually not given and the equations of motion are traditionally solved in terms of elementary functions (hyperelliptical in the case of Kovalevskaya and elliptical in the three other cases). ■

In Section 4.11 we give a method to build simple Jacobi multipliers for a large class of vector fields (for another method see Gorbusov (1994)).

2.12 Lax pairs

One of the most mysterious objects appearing in the theory of differential equations is the Lax pair. A "Lax pair" is a couple of linear operators whose coefficients depend on $\mathbf{x} = \mathbf{x}(t)$ and whose compatibility condition is the vector field itself. Whereas the concept of Lax pairs can be traced back to the last century, Peter Lax (1968) was the first to define properly the general concept in the context of evolution equation (and more particularly for the Korteweg-de Vries (KdV) equation). Since then they have become a central object in the analysis of integrable systems. The discussion of Lax pairs in this section will be at the most elementary level; we will merely be interested in their general algebraic properties. A discussion of Lax pairs in the context of algebraic geometry can be found for instance in a delightful little book by Audin (1996).

2.12.1 General properties

To define a Lax pair for a system of ODEs, we first extend the definition of a derivation to matrices (Churchill & Falk, 1995). Consider an $m \times m$ matrix A whose entries are C^1 functions of $\mathbf{x} = (x_1, \dots, x_n)$, that is, $A \in \mathcal{M}_m(C^1(\mathbf{x}))$. The *time derivative* $\dot{A}$ *of* A is defined as the matrix whose entries are the time derivative of the entries of A along the vector field $\mathbf{f}$. That is,

$$(\dot{A})_{ij} = \dot{A}_{ij} = \sum_{k=1}^{n} f_k \partial_{x_k} A_{ij}. \tag{2.148}$$

That is,

$$\dot{A} = \delta_{\mathbf{f}} A = \mathbf{f}.\partial_{\mathbf{x}} A. \tag{2.149}$$

Definition 2.21 A *Lax pair* for the vector field $\delta_{\mathbf{f}}$ is an ordered pair of matrices (A, B), $A, B \in \mathcal{M}_m(C^1(\mathbf{x}))$ such that

$$\dot{A} = [B, A], \tag{2.150}$$

where $[B, A] = BA - AB$. The Lax pair is *non-trivial* if $\dot{A} \neq (\mathbf{0})$ and *trivial* otherwise.

This definition is best understood on an example.

Example 2.28 A simple Lax pair. The system

$$\dot{x} = x(y - z), \tag{2.151.a}$$
$$\dot{y} = y(z - x), \tag{2.151.b}$$
$$\dot{z} = z(x - y). \tag{2.151.c}$$

admits the Lax pair

$$A = \begin{bmatrix} 0 & 1 & x \\ y & 0 & 1 \\ 1 & z & 0 \end{bmatrix}, \quad B = \begin{bmatrix} x+y & 0 & 1 \\ 1 & y+z & 0 \\ 0 & 1 & x+z \end{bmatrix}. \tag{2.152}$$

We verify by direct computation that

$$\begin{aligned} \dot{A} &= \begin{bmatrix} 0 & 0 & \dot{x} \\ \dot{y} & 0 & 0 \\ 0 & \dot{z} & 0 \end{bmatrix}, \\ &= \begin{bmatrix} 0 & 0 & x(y-z) \\ y(z-x) & 0 & 0 \\ 0 & z(x-y) & 0 \end{bmatrix}, \\ &= \begin{bmatrix} x+y & 0 & 1 \\ 1 & y+z & 0 \\ 0 & 1 & x+z \end{bmatrix} \begin{bmatrix} 0 & 1 & x \\ y & 0 & 1 \\ 1 & z & 0 \end{bmatrix} - \\ & \qquad \begin{bmatrix} 0 & 1 & x \\ y & 0 & 1 \\ 1 & z & 0 \end{bmatrix} \begin{bmatrix} x+y & 0 & 1 \\ 1 & y+z & 0 \\ 0 & 1 & x+z \end{bmatrix} \\ &= BA - AB. \end{aligned} \tag{2.153}$$

■

We want to impose another restriction on the Lax pair. We want the Lax pair to contain all the information regarding the original vector field. That is, we want to be able to obtain the complete vector field **f** from the Lax pair relation (2.150). A Lax pair is referred to as *regular* if **f** can be found from (2.150) and *irregular* otherwise. More precisely, $\dot{A}$ defines m^2 linear relations on $\dot{\mathbf{x}}$: $\partial_{x_k} A_{ij} \dot{x}_k$. If there exists a domain U such that for all $\mathbf{x} \in U$, n of these relations are linearly independent, then the Lax pair is regular.

Example 2.29 An irregular Lax pair. Consider the system from the previous example with an added equation

$$\dot{x} = x(y-z), \tag{2.154.a}$$
$$\dot{y} = y(z-x), \tag{2.154.b}$$
$$\dot{z} = z(x-y), \tag{2.154.c}$$
$$\dot{w} = f(x,y,z,w), \tag{2.154.d}$$

then, the Lax pair (2.152) for the system (2.151) is still, according to our definition, a Lax pair for this four-dimensional system. However, the vector field cannot be recovered from the Lax pair since A does not depend on the variable w and there exist only 3 non-zero entries of $\dot{A}$. Hence, the Lax pair (2.152) is irregular for (2.154). ■

The importance of Lax pairs for the theory of integrability lies in the following remarkable property.

Proposition 2.12 *Let (A, B) be a Lax pair for $\delta_{\mathbf{f}}$, then $\mathrm{Tr}(A^k)$ is a first integral of $\delta_{\mathbf{f}}$ for all $k \in \mathbb{N}$.*

Proof. We show that the time derivative of $\mathrm{Tr}(A^k)$ vanishes identically. To do so, we repeatedly use the properties of the trace: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ and $\mathrm{Tr}(A+B) = \mathrm{Tr}(B+A)$

$$\begin{aligned}
\frac{d}{dt}\mathrm{Tr}(A^k) &= \mathrm{Tr}(\frac{dA^k}{dt}),\\
&= \mathrm{Tr}(\dot{A}A^{k-1} + A\dot{A}A^{k-2} + \ldots + A^{k-1}\dot{A}),\\
&= k\mathrm{Tr}(\dot{A}A^{k-1}),\\
&= k\mathrm{Tr}(BA^k - ABA^{k-1}),\\
&= k\mathrm{Tr}(BA^k - BA^k),\\
&= k\mathrm{Tr}(0),\\
&= 0. \qquad (2.155)
\end{aligned}$$

□

An interesting corollary of this proposition is that the eigenvalues of A are also constant.

Corollary 2.3 *Let (A, B) be a Lax pair for $\delta_{\mathbf{f}}$, then λ is a first integral of $\delta_{\mathbf{f}}$ for all $\lambda \in \mathrm{Spec}(A)$.*

Proof. The coefficients of the characteristic polynomial of a given matrix are symmetric functions of the eigenvalues $(\lambda_1, \ldots, \lambda_n)$. We also have: $\mathrm{Tr}(A^k) = \sum_{i=1}^n \lambda_i^k$. Hence, $\mathrm{Tr}(A^k)$ generates all the symmetric functions of the eigenvalues and the coefficients of the characteristic polynomial are first integrals. The eigenvalues are therefore algebraic functions of the first integrals $\mathrm{Tr}(A^k)$. □

Example 2.30 First integrals. We can now revisit Example 2.28 and compute the first integrals of system (2.151) by computing the traces of increasing powers of A:

$$\mathrm{Tr}(A) = 0, \qquad (2.156.a)$$
$$\mathrm{Tr}(A^2) = 2(x+y+z), \qquad (2.156.b)$$
$$\mathrm{Tr}(A^3) = 3(1+xyz), \qquad (2.156.c)$$
$$\mathrm{Tr}(A^4) = 2(x+y+z)^2, \qquad (2.156.d)$$
$$\mathrm{Tr}(A^5) = 5(x+y+z)(1+xyz). \qquad (2.156.e)$$

We conclude that $I_1 = x+y+z$ and $I_2 = xyz$ are first integrals. ■

The last example clearly shows that for an $m \times m$ matrix depending on n variables, only n of the m traces can be functionally independent (since the traces depend on the $n \leq m$ eigenvalues). However, the traces of increasing powers do not always give all the first integrals of the system.

Example 2.31 The system

$$\dot{x} = (\mu_3 - \mu_2)yz, \qquad (2.157.a)$$
$$\dot{y} = (\mu_1 - \mu_3)xz, \qquad (2.157.b)$$
$$\dot{z} = (\mu_2 - \mu_1)xy, \qquad (2.157.c)$$

has the regular Lax pair

$$A = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \mu_3 z & -\mu_2 y \\ -\mu_3 z & 0 & \mu_1 x \\ \mu_2 y & -\mu_1 x & 0 \end{bmatrix}. \qquad (2.158)$$

The different traces of A^k give $\mathrm{Tr}(A) = \mathrm{Tr}(A^3) = 0$ and $\mathrm{Tr}(A^2) = -2I_1 = -2(x^2+y^2+z^2)$. However, the system admits another first integral, $I_2 = \mu_1^2 x^2 + \mu_2^2 y^2 + \mu_3^2 z^2$ that cannot be found directly from the matrix A. Note, however that, strangely enough, $\mathrm{Tr}(AB) = 2I_2$. ■

An interesting question is whether the missing first integrals can be found from the Lax pair. We will return to this problem in Section 2.12.2. The Lax pair formalism is also a very important tool to study hierarchies of integrable systems. These systems are n-dimensional systems defined for arbitrary values of n. They typically arise as reductions of PDEs.

Example 2.32 The Toda system. Toda (1967) derived a simple model for the dynamics of interacting particles with repulsive exponential forces (Toda, 1967; Toda, 1981). Its Hamiltonian reads

$$H = \sum_{i=1}^{N} \left[\frac{p_i^2}{2} + e^{q_i - q_{i+1}} + \mu(q_i - q_{i+1}) \right]. \tag{2.159}$$

The particles evolve on a circle, so that $q_{N+1} = q_1$. The Hamilton equations are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \ldots, N. \tag{2.160}$$

For $\mu = 0$, (A, B) is a Lax pair with

$$A = \begin{bmatrix} b_1 & a_1 & 0 & \ldots & 0 & a_N \\ a_1 & b_2 & a_2 & \ldots & 0 & 0 \\ 0 & a_2 & b_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{N-1} & a_{N-1} \\ a_N & 0 & 0 & \ldots & a_{N-1} & b_N \end{bmatrix}, \tag{2.161.a}$$

$$B = \begin{bmatrix} 0 & a_1 & 0 & \ldots & 0 & -a_N \\ -a_1 & 0 & a_2 & \ldots & 0 & 0 \\ 0 & -a_2 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & a_{N-1} \\ a_N & 0 & 0 & \ldots & -a_{N-1} & 0 \end{bmatrix} \tag{2.161.b}$$

and

$$a_i = \frac{1}{2} \exp\left(\frac{q_i - q_{i+1}}{2} \right), \qquad b_i = -\frac{p_i}{2}. \tag{2.162}$$

This Lax pair was first discovered independently by H. Flaschka and S. V. Manakov in 1974 (Flaschka, 1974; Flaschka, 1975; Manakov, 1975). This general form for the Lax pair allows us to compute many properties of the Toda system. For instance for $N = 3$ and $\mu = 0$, the Hamiltonian reads

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + X_1 + X_2 + X_3, \tag{2.163}$$

with $X_i = \exp(q_i - q_{i+1})$ and three first integrals can be found from the Lax pairs

$$\begin{aligned} I_1 &= H, \\ I_2 &= p_1 + p_2 + p_3, \\ I_3 &= \frac{1}{3}(p_1^3 + p_2^3 + p_3^3) + p_1(X_1 + X_2) + p_2(X_2 + X_3) + p_3(X_3 + X_1). \end{aligned} \tag{2.164}$$

Furthermore, the Lax pair can be used to explicitly integrate the equations of the motion and find most of the general properties of the solutions (such as asymptotic behavior, periodic solutions). See, for instance, the book by Perelomov (1990). ■

As already mentioned, the Lax pair equation can be seen as the compatibility condition between two linear systems. That is, the Lax pair is the compatibility condition arising from the commutation of a pair of linear operators.

Proposition 2.13 *Let A be a semi-simple*[9]*, then the Lax pair equation (2.150) is the compatibility condition between the two linear operators L_1, L_2 defined by the equations*

$$L_1\Psi = (A - \lambda\mathbf{1})\Psi = 0, \tag{2.165.a}$$

$$L_2\Psi = (\partial_t - B)\Psi = 0, \tag{2.165.b}$$

where $\Psi = \Psi(t) \in (C^1(t))^n$, A and B are functions of $\mathbf{x} = \mathbf{x}(t) \in (C^1(t))^n$ and λ is a constant.

Proof. The compatibility condition between the operators L_1 and L_2 is an equation for $\mathbf{x}$ such that the two linear equations $L_1\Psi = 0$ and $L_2\Psi = 0$ have common solutions. To find such an equation, we compute $L_1(L_2(\Psi)) - L_2(L_1(\Psi)) = 0$ or, equivalently, we take the time-derivative of (2.165.a) and use (2.165.b) to simplify it to obtain

$$\dot{A}\Psi + A\dot{\Psi} = \lambda\dot{\Psi}. \tag{2.166}$$

That is,

$$\dot{A}\Psi + AB\Psi = \lambda B\Psi = B\lambda\Psi = BA\Psi. \tag{2.167}$$

Since A is semi-simple, the set of eigenvectors Ψ of A forms a complete basis and

$$\dot{A}\Psi = (BA - AB)\Psi, \tag{2.168}$$

implies $\dot{A} = [B, A]$. □

These pairs of linear operators do not always come in a matrix form. For instance, in the theory of PDEs, the Lax pairs are often represented by a pair of linear operators of nth order in one variable. A natural question is then: how can we write such a pair of linear operators in the form (A, B)? This problem is similar to the problem of transforming an nth order differential equation to a system of n first order ODEs.

[9] A matrix A is *semi-simple* matrix if there exists an invertible matrix B such that $B^{-1}AB$ is a diagonal matrix. That is, matrix A can be diagonalized.

Example 2.33 The procedure to build the matrices (A, B) from a pair of linear operators can be illustrated by considering the simple system

$$\dot{x} = y, \tag{2.169.a}$$
$$\dot{y} = -3x^2. \tag{2.169.b}$$

This system has a pair of Lax operators given by

$$L_1 = \frac{\partial^2}{\partial t^2} + x - \lambda, \tag{2.170.a}$$
$$L_2 = 4\frac{\partial^3}{\partial t^3} + 6x\frac{\partial}{\partial t} + 3y - \mu. \tag{2.170.b}$$

The commutation relation $(L_1L_2 - L_2L_1)\Psi = 0$ is identically satisfied on the derivative of the equation, that is, when $\partial_t(\partial_{tt}x + 3x^2) = 0$. A typical feature of a pair of Lax operators is that one of the operators is of higher order. For instance, in our case L_2 is of order 3. We can then introduce the variables: $\varphi_1 = \Psi,\ \varphi_2 = \dot{\varphi}_1,\ \varphi_3 = \dot{\varphi}_2$. These 3 relations together with $L_2\Psi = 0$ can be written

$$\begin{aligned} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\varphi}_3 \end{bmatrix} &= \begin{bmatrix} \varphi_2 \\ \varphi_3 \\ \frac{\mu-3y}{4}\varphi_1 - \frac{3}{2}x\varphi_2 \end{bmatrix}, \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\mu-3y}{4} & -\frac{3}{2}x & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}. \end{aligned} \tag{2.171}$$

This provides matrix B from the relation (2.165.b). To obtain A, we consider the equation $L_1\Psi = 0$ and its derivatives. Whenever the third derivative of Ψ appears, we use the relation $L_2\Psi = 0$ to obtain

$$\begin{aligned} &\begin{bmatrix} (x-\lambda)\varphi_1 + \varphi_3 \\ \frac{y+\mu}{4}\varphi_1 - (\frac{x}{2} + \lambda)\varphi_2 \\ -\frac{3}{4}x^2\varphi_1 + \frac{\mu-y}{4}\varphi_2 - (\frac{x}{2} + \lambda)\varphi_3 \end{bmatrix} = \\ &\begin{bmatrix} x-\lambda & 0 & 1 \\ \frac{y+\mu}{4} & -\frac{x}{2} - \lambda & 0 \\ -\frac{3}{4}x^2 & \frac{\mu-y}{4} & -\frac{x}{2} - \lambda \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{2.172}$$

This provides matrix A from the relation (2.165.a). We can now compute the trace of A^k to obtain the first integral of the system

$$\mathrm{Tr}(A) = \mathrm{Tr}(A^2) = 0, \quad \mathrm{Tr}(A^3) = -\frac{3}{8}\left(x^3 + \frac{y^2}{2} - \mu^2\right). \tag{2.173}$$

Note that the extra parameter μ is superfluous in the computation of the first integral. This so-called "spectral parameter" plays, however, a crucial role in the determination of the geometric properties of the solution. ■

It is clear from the last example that this type of construction can be done in general and that a pair of linear operators of order n and $(n - l)$ $(l > 0)$ can be written as a pair made of one system of n first order equations and a system of n linear equations.

There is yet another way to represent matrix Lax pairs. To do so, we consider the matrices A and B as functions of an arbitrary constant λ.

Example 2.34 A reduced fifth order KdV equation. The four-dimensional system

$$\dot{x} = y, \tag{2.174.a}$$
$$\dot{y} = u - \frac{5}{2}x^2, \tag{2.174.b}$$
$$\dot{u} = v, \tag{2.174.c}$$
$$\dot{v} = \frac{5}{2}x^3 - 5uv, \tag{2.174.d}$$

can be obtained from the stationary reduction of the fifth order stationary Korteweg-de Vries equation (Blaszak & Rauch-Wojciechowski, 1994)

$$U_t + \left(U_{4x} + 10UU_{xx} + 5U_x^2 + 10U^3\right)_x = 0. \tag{2.175}$$

Therefore, the Lax pair for (2.174) can be obtained from the Lax pair of KdV. In this case, when written as a matrix Lax pair, we have

$$A = \begin{bmatrix} -4\lambda y - v - xy & 16\lambda^2+8\lambda x+2u+x^2 \\ 16\lambda^3-8\lambda^2x+3\lambda x^2-2\lambda u-y^2-x^3+2xu & 4\lambda y + v + xy \end{bmatrix}, \tag{2.176.a}$$

$$B = \begin{bmatrix} 0 & 1 \\ \lambda - x & 0 \end{bmatrix}. \tag{2.176.b}$$

The determinant of A contains two first integrals

$$\det(A) = -256\lambda^5 + I_1\lambda - I_2, \tag{2.177}$$

where

$$I_1 = -20x^2u + 5x^4 + 4u^2 - 8yv, \tag{2.178.a}$$
$$I_2 = v^2 + 2xyv - 2y^2u + 4xu^2 - x^5. \tag{2.178.b}$$

Since the vector field (2.174) does not depend on λ, the coefficients of all constant polynomials of λ are themselves first integrals. Hence, we conclude that I_1 and I_2 are first integrals. ■

We will refer to these Lax pairs as λ-*Lax pairs* and write $(A(\lambda), B(\lambda))$. This representation allows us to write a Lax pair in a compact form and uses the following property to compute the first integrals.

Proposition 2.14 *Let $(A(\boldsymbol{\lambda}), B(\boldsymbol{\lambda}))$ be a Lax pair of $\delta_{\mathbf{f}}$ where A and B are polynomials in $\boldsymbol{\lambda} \in \mathbb{K}^k$ and $\mathbf{f}$ is independent of $\boldsymbol{\lambda}$, then we have $\det(A) = \sum_{\mathbf{j},|\mathbf{j}|=0}^{l} I_{\mathbf{j}}\boldsymbol{\lambda}^{\mathbf{j}}$ for some $l \in \mathbb{N}$ and all the coefficients $I_{\mathbf{j}}$ are first integrals of $\mathbf{f}$.*

Proof. The matrix A is a polynomial in $\boldsymbol{\lambda}$, hence so is its determinant. Since the eigenvalues of A are first integrals, so is $\det(A)$. The vector field does not depend on $\lambda_1, \ldots, \lambda_k$. Therefore, every coefficient vanishes separately. □

These λ-Lax pairs can be written in terms of a Lax pair with no extra arbitrary parameter.

Proposition 2.15 *Let $(A(\lambda), B(\lambda))$ be a λ-Lax pair for $\delta_{\mathbf{f}}$ where $A = \sum_{i=0}^{a} A_i\lambda^i$ and $B = \sum_{i=0}^{b} B_i\lambda^i$. Then, $(\mathcal{A}, \mathcal{B})$ is a Lax pair for $\delta_{\mathbf{f}}$ with*

$$\mathcal{A}=\begin{bmatrix} A_0 & A_1 & \dots & A_a & 0 & \dots & 0 & 0 & 0 \\ 0 & A_0 & A_1 & \dots & A_a & 0 & \dots & 0 & 0 \\ 0 & 0 & A_0 & A_1 & \dots & A_a & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & A_0 & A_1 & A_2 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & A_0 & A_1 & A_2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & A_0 & A_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & A_0 \end{bmatrix},$$

$$\mathcal{B}=\begin{bmatrix} B_0 & B_1 & \dots & B_b & 0 & \dots & 0 & 0 & 0 \\ 0 & B_0 & B_1 & \dots & B_b & 0 & \dots & 0 & 0 \\ 0 & 0 & B_0 & B_1 & \dots & B_b & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & B_0 & B_1 & B_2 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & B_0 & B_1 & B_2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & B_0 & B_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & B_0 \end{bmatrix}, \tag{2.179}$$

and dim$(\mathcal{A})$ =dim$(\mathcal{B})$ =dim$(A)(a+b-1)$.

Proof. The Lax pair relation (2.150) reads

$$\sum_{i=1}^{a} \dot{A}_i\lambda^i = \left[\sum_{i=1}^{b} \dot{B}_i\lambda^i, \sum_{i=1}^{a} \dot{A}_i\lambda^i\right]. \tag{2.180}$$

Since the vector field is independent of λ, this relation can be decoupled into the set of linear equations

$$\dot{A}_i = \sum_{j=0}^{i} [B_j, A_{i-j}], \qquad i = 0, \dots, a+b, \tag{2.181}$$

where $A_j = B_k = 0$ for all $j > a, k > b$. A direct computation shows that this set of relations is equivalent to the conditions obtained from

$$\dot{\mathcal{A}} = [\mathcal{B}, \mathcal{A}]. \tag{2.182}$$

□

Lax pairs are not unique. A given system can have many Lax pairs. For instance, assume that (A, B) is a Lax pair for a vector field $\delta_{\mathbf{f}}$, then another Lax pair for $\delta_{\mathbf{f}}$ is given by $(A + J(\mathbf{x})\mathrm{I}, B)$, where $J(\mathbf{x})$ is a first integral. In general, infinitely many Lax pairs can be built starting from a given Lax pair.

Proposition 2.16 *Let (A, B) be a Lax pair for $\delta_{\mathbf{f}}$ and let P be an analytic function. Then, $(P(A), B)$ is another Lax pair for $\delta_{\mathbf{f}}$.*

Proof. Let (A, B) be a Lax pair for $\delta_{\mathbf{f}}$. We show that *(i)* $(\gamma A^k, B)$ is a Lax pair for all $k \in \mathbb{N}, \gamma \in \mathbb{K}$, and *(ii)* $(A + \tilde{A}, B)$ is a Lax pair for all $\tilde{A}$ such that $(\tilde{A}, B)$ is a Lax pair. The result follows directly. *(i)* To show that $(\gamma A^k, B)$ is a Lax pair, we compute $\dot{\overline{\gamma A^k}}$ explicitly

$$
\begin{aligned}
\dot{\overline{\gamma A^k}} &= \gamma \sum_{i=0}^{k-1} A^i \dot{A} A^{k-1-i}, \\
&= \gamma \sum_{i=0}^{k-1} A^i [B, A] A^{k-1-i}, \\
&= \gamma \sum_{i=0}^{k-1} A^i (BA - AB) A^{k-1-i}, \\
&= \gamma \left(\sum_{i=0}^{k-1} A^i B A^{k-i} - \sum_{i=0}^{k-1} A^{i+1} B A^{k-i} \right), \\
&= \gamma \left(BA^k - BA^k \right), \\
&= [B, \gamma A^k]. \qquad (2.183)
\end{aligned}
$$

(ii) Second, we compute the derivative of $A + \tilde{A}$ assuming that both (A, B) and $(\tilde{A}, B)$ are Lax pairs: $\dot{\overline{A + \tilde{A}}} = BA - AB + B\tilde{A} - \tilde{A}B = [B, A + \tilde{A}]$. □

In the same way, the choice of matrix B in a Lax pair of a given system is not unique.

Proposition 2.17 *Let (A, B) be a Lax pair for $\delta_{\mathbf{f}}$, then $(A, \tilde{B})$, is another Lax pair for $\delta_{\mathbf{f}}$ if and only if $[B - \tilde{B}, A] = 0$.*

Proof. This follows directly form the definition of a Lax pair: $\dot{A} = [B, A] = [\tilde{B}, A]$. Hence, $[B - \tilde{B}, A] = 0$. □

2.12.2 Construction of Lax pairs

To date, there is no general way of finding Lax pairs without knowing the first integrals. In some cases, the vector field can be obtained as the reduction of a PDE. The Lax pair for the vector field is then the Lax pair obtained by the corresponding reduction. For instance, this is the case for all the ODEs obtained as the reduction of integrable hierarchies (such as the Korteweg-de Vries). In other cases, the Lie group structures of the equations provide a direct way to build the Lax pair of an integrable system (Adler, 1979; Kostant, 1979; Semenov-Tian-Shanski, 1983). This is how the Lax pairs for the Euler equations (in the known integrable cases) have been obtained. A third way to obtain Lax pairs is to use the properties of the local series expansions around the singularities. The general construction and analysis of the local solutions around their movable singularities is performed in the next chapter. Unfortunately, the construction of Lax pairs by this last method is not yet thoroughly understood and only works in very particular cases (strangely enough, the method seems to work mostly for PDEs). However, there is a growing body of literature on the subject and we can hope that a general method to build Lax pairs from the local solutions will be developed (Weiss, 1983; Weiss, 1985a; Tabor & Gibbon, 1986; Conte, 1988; Newell *et al.*, 1987; Musette & Conte, 1991; Flaschka *et al.*, 1991; Estevéz & Gordoa, 1997; Estevéz *et al.*, 1998; Estevéz & Gordoa, 1998; Estevéz, 1999).

There is also no general, algorithmic way to build directly a Lax pair for an algebraically integrable system. Is it possible to build a Lax pair for a system once some of the first integrals are known? We outline in this section three different methods to find Lax pairs for a vector field with given first integrals. The first method is due to Churchill and Falk (1995).

Proposition 2.18 *Let $I_1, \dots, I_k$ be k independent first integrals of $\delta_{\mathbf{f}}$, then the pair of matrices*

$$A = \begin{bmatrix} g(\mathbf{x}) & 1 \\ -g(\mathbf{x}) - \sum_{i=1}^{k} I_i \lambda_i & -g(\mathbf{x}) \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -\mathbf{f}.\partial_{\mathbf{x}} g & 0 \end{bmatrix}, \tag{2.184}$$

where $\lambda_1, \dots, \lambda_k$ are arbitrary parameters and $g(\mathbf{x})$ is any non constant function of $\mathbf{x}$, is a Lax pair.

Proof. By direct computation, we can verify that $(A(\lambda), B(\lambda))$ is a λ-Lax pair. The determinant of A is the first integral $\det(A) = \sum_{i=1}^{k} I_i \lambda_i$ and each coefficient of λ_i is a first integral. □

However, the Lax pairs built in this way may not be regular since the Lax pair may not be enough to recover the vector field.

Proposition 2.19 *Let $I_1, \dots, I_k$ be k independent first integrals of $\delta_{\mathbf{f}}$. Consider the matrix $S =$ $\mathrm{diag}(I_1, \dots, I_k, \lambda_{k+1}, \dots, \lambda_m)$ where $\lambda_{k+1}, \dots, \lambda_m$ are arbitrary parameters and $M = M(\mathbf{x}) \in \mathrm{SL}(m, \mathbb{K})\ \forall\, \mathbf{x}$. Then, the pair of matrices*

$$A = MSM^{-1}, \quad B = \dot{M}M^{-1}, \tag{2.185}$$

is a Lax pair for $\delta_{\mathbf{f}}$.

The choice $\det(M) = 1$ ensures that if all the first integrals are polynomial, the Lax pair is also polynomial.

Proof. By construction, we have $\dot{S} = 0$ and Spec(A) =Spec(S). We must check that (A, B) is a Lax pair.

$$\begin{aligned} \frac{d}{dt}A &= \dot{M}SM^{-1} + M\dot{S}M^{-1} - MSM^{-1}\dot{M}M^{-1}, \\ &= \dot{M}SM^{-1} - MSM^{-1}\dot{M}M^{-1}, \\ &= \dot{M}M^{-1}MSM^{-1} - MSM^{-1}\dot{M}M^{-1}, \\ &= [\dot{M}M^{-1}, MSM^{-1}], \\ &= [B, A]. \end{aligned}$$

□

Another way to tackle the problem is to find a general ansatz for Lax pairs for a large class of systems (Mumford, 1984; Churchill & Falk, 1995).

Proposition 2.20 *Consider two functions $g = g(\mathbf{x})$ and $h = h(\mathbf{x})$ where $\mathbf{x}(t)$ is a solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ such that $\dot{g} \neq 0$. Assume that the functions g and h are chosen such that*

$$\frac{d^3 g}{dt^3} = 4h\frac{dg}{dt} + 2\frac{dh}{dt}g, \tag{2.186}$$

then the pair of matrices

$$A = \begin{bmatrix} -\dot{g} & 2g \\ -\ddot{g} + 2gh & \dot{g} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ h & 0 \end{bmatrix}, \tag{2.187}$$

is a Lax pair for the vector field $\delta_{\mathbf{f}}$.

Proof. A direct computation of the Lax relation gives the result and is left as an exercise. □

The problem to find a suitable pair (g, h) of non-constant functions such that (2.186) holds can be solved by taking any function $s = s(\mathbf{x})$ and setting $g = s^2$ and $h = \ddot{s}/s$.

Example 2.35 The Hénon-Heiles Hamiltonian. The Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(aq_1^2 + bq_2^2) + \frac{1}{3}q_1^3 + cq_1q_2^2, \tag{2.188}$$

is known to be integrable in three cases:

$$\text{Case I}: \quad (a, b, c) = (a, b, 1/6), \tag{2.189}$$
$$\text{Case II}: \quad (a, b, c) = (a, a, 1), \tag{2.190}$$
$$\text{Case III}: \quad (a, b, c) = (a, 0, 0). \tag{2.191}$$

In order to find the Lax pair, we set

$$g = \lambda^2 + g_1\lambda + g_0, \tag{2.192}$$
$$h = \lambda^2 + h_0, \tag{2.193}$$

where g_1, g_0, h_0 are polynomial in $\mathbf{x} = (q_1, q_2, p_1, p_2)$ with undetermined coefficients chosen such that (2.186) is satisfied. A tedious computation gives the following results

1. $(a, b, c) = (a, b, 1/6)$:

$$g_1 = \frac{q_1}{6}, \tag{2.194}$$
$$g_0 = -\frac{(4b-a)^2}{64} + \frac{1}{48}(4b-a)q_1 - \frac{q_2^2}{144} \tag{2.195}$$
$$h_0 = -\frac{q_1}{3} - \frac{1}{8}(4b+a), \tag{2.196}$$

and the computation of the determinant of a provides the second first integral

$$G = \frac{3}{2}(4b-a)(bq_2^2+p_2^2)+bq_1q_2^2 + p_2(q_2p_1 - q_1p_2) + \frac{1}{6}q_2^2(q_1^2+\frac{1}{4}q_2^2). \tag{2.197}$$

2. $(a, b, c) = (a, a, 1)$:

$$g_1 = \frac{1}{6}(q_1 + q_2 + \gamma H), \tag{2.198}$$
$$g_0 = \frac{a - 2\gamma H}{48}(2q_1 + 2q_2 - 4\gamma H + 3a), \tag{2.199}$$
$$h_0 = -\frac{q_1 + q_2 + \gamma H}{3}, \tag{2.200}$$

where γ is arbitrary. If $\gamma = 0$, the determinant of A does not provide an independent first integral. However, for $\gamma \neq 0$, we have

$$G = p_1p_2 + q_1p_2(a + q_1) + \frac{1}{3}q_2^2. \tag{2.201}$$

3. $(a,b,c)=(a,0,0)$:

$$g_1 = \frac{1}{6}q_1 + \frac{a}{12}, \qquad g_0 = 0, \tag{2.202}$$

$$h_0 = -\frac{q_1}{3} + \frac{a}{6}, \tag{2.203}$$

here again, we find an additional first integral:

$$G = p_2. \tag{2.204}$$

It is worth noting that the computation involved in finding the Lax pair does not seem to be any shorter than a direct computation of the polynomial first integrals. ■

2.12.3 Completion of Lax pairs

We saw in the previous sections that it is not always possible to obtain all the first integrals from the Lax pairs. However, it might be possible to build new Lax pairs from an "incomplete" Lax pair. A possible way of completing a Lax pair is to add a pair of arbitrary matrices $(\tilde{A}, \tilde{B})$ to (A, B), that is, we define the new pair of matrices to be

$$A_1(\lambda) = A + \lambda\tilde{A}, \tag{2.205.a}$$

$$B_1(\lambda) = B + \lambda\tilde{B}. \tag{2.205.b}$$

The Lax relation for (A_1, B_1) is

$$\dot{A}_1 = [B_1, A_1]. \tag{2.206}$$

That is,

$$\dot{A} + \lambda\dot{\tilde{A}} = [B, A] + \lambda([\tilde{B}, A] + [B, \tilde{A}]) + \lambda^2[\tilde{B}, \tilde{A}]. \tag{2.207}$$

Hence, we have

$$\dot{\tilde{A}} = [\tilde{B}, A] + [B, \tilde{A}], \tag{2.208.a}$$

$$0 = [\tilde{B}, \tilde{A}]. \tag{2.208.b}$$

This system cannot be solved in general. However, there are two simple choices of matrices that satisfy the commutation relation: *(i)* $\tilde{B} = u(\mathbf{x})\tilde{A}$, or *(ii)* $\tilde{A} = \mathrm{diag}(a_1, \dots, a_m)$, $\tilde{B} = \mathrm{diag}(b_1, \dots, b_m)$. The first relation gives an equation for the entries of A, that is, for the components of the vector $\mathbf{a}$ and $\mathbf{b}$.

Example 2.36 Consider again system (2.157)

$$\dot{x} = (\mu_3 - \mu_2)yz, \tag{2.209.a}$$

$$\dot{y} = (\mu_1 - \mu_3)xz, \tag{2.209.b}$$

$$\dot{z} = (\mu_2 - \mu_1)xy. \tag{2.209.c}$$

If we choose $\tilde{A} = \mathrm{diag}(a_1, a_2, a_3)$ and $\tilde{B} = \mathrm{diag}(b_1, b_2, b_3)$ with

$$\mathbf{a} = \frac{1}{2}(\mu_1 - \mu_2 - \mu_3, -\mu_1 + \mu_2 - \mu_3, -\mu_1 - \mu_2 + \mu_3), \tag{2.210.a}$$

$$\mathbf{b} = (a_1^2, a_2^2, a_3^2), \tag{2.210.b}$$

we obtain

$$\mathrm{Tr}(A_1^2) = \lambda^2 C_1 - 2I_1, \tag{2.211.a}$$

$$\mathrm{Tr}(A_1^2) = \lambda^3 C_2 - 3\lambda I_2, \tag{2.211.b}$$

where $I_1 = x^2 + y^2 + z^2$ and $I_2 = \mu_1^2 x^2 + \mu_2^2 y^2 + \mu_3^2 z^2$ and C_1, C_2 are constants (Steeb & van Tonder, 1988a). ■

In the case where some of the eigenvalues of A and B are degenerate, some of the coefficients of the eigenvectors Ψ can be constant.

Proposition 2.21 *Let (A, B) be a Lax pair for $\delta_{\mathbf{f}}$. Assume that μ is a constant eigenvalue of B with eigenspace $\boldsymbol{\beta}_\mu$. If there exists an eigenvalue λ of A with eigenspace $\boldsymbol{\alpha}_\lambda$ such that $0 \neq \Psi \in \boldsymbol{\alpha}_\lambda \cap \boldsymbol{\beta}_\mu$, then $\Psi_i e^{-\mu t} = I_i$ are first integrals of $\delta_{\mathbf{f}}$ and the number of independent first integrals is m=dim$(\boldsymbol{\alpha}_\lambda \cap \boldsymbol{\beta}_\mu)$.*

Proof. Consider relation (2.165)

$$A\Psi = \lambda\Psi, \tag{2.212.a}$$

$$B\Psi = \dot{\Psi}. \tag{2.212.b}$$

If there exists a non-zero vector $\Psi \in \boldsymbol{\alpha}_\lambda \cap \boldsymbol{\beta}_\mu$, then $B\Psi = \mu\Psi = \Psi_t$ and $\Psi_i e^{-\mu t} = I_i$, $i = 1, \ldots, m$. □

This proposition is especially useful when both A and B have large kernels[10] with non-zero intersections.

Example 2.37 Ocean wave interactions. The following system describes the restricted multiple three wave interactions where interacting waves are coupled with equal strength. It has been used to model the interaction of a low-frequency internal ocean waves with higher-frequency surface waves (Menyuk *et al.*, 1982). The system reads

$$\dot{x}_0 = i \sum_{k=1}^{N} x_k y_k^*, \tag{2.213.a}$$

$$\dot{x}_k = i x_0 y_k, \qquad k = 1, \ldots, N, \tag{2.213.b}$$

$$\dot{y}_k = i x_0^* x_k, \qquad k = 1, \ldots, N. \tag{2.213.c}$$

These equations are the Hamilton equations for the Hamiltonian

$$H = i \sum_{k=1}^{N} \left(x_0 x_k^* y_k + x_0^* x_k y_k^* \right), \tag{2.214}$$

where each variable is canonically conjugate to its complex conjugate. The Lax pair is

$$A = \begin{bmatrix} 0 & -x_0 & -\frac{x_1}{2} & -\frac{y_1^*}{2} & \ldots & -\frac{x_N}{2} & -\frac{y_N^*}{2} \\ x_0^* & 0 & -\frac{y_1}{2} & \frac{x_1^*}{2} & \ldots & -\frac{y_N}{2} & \frac{x_N^*}{2} \\ \frac{x_1^*}{2} & -\frac{y_1^*}{2} & 0 & 0 & \ldots & 0 & 0 \\ -\frac{y_1}{2} & -\frac{x_1}{2} & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ \frac{x_N^*}{2} & -\frac{y_N^*}{2} & 0 & 0 & \ldots & 0 & 0 \\ -\frac{y_N}{2} & -\frac{x_N}{2} & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{2.215}$$

[10]The *Kernel* of a matrix A is the vector space spanned by the vectors that are sent to the origin. That is, $\mathrm{Ker}(A) = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\}$.

$$B = \begin{bmatrix} 0 & 0 & i\frac{x_1}{2} & i\frac{y_1^*}{2} & \dots & i\frac{x_N}{2} & i\frac{y_N^*}{2} \\ 0 & 0 & -i\frac{y_1}{2} & i\frac{x_1^*}{2} & \dots & -i\frac{y_N}{2} & i\frac{x_N^*}{2} \\ i\frac{x_1^*}{2} & -i\frac{y_1^*}{2} & 0 & 0 & \dots & 0 & 0 \\ i\frac{y_1}{2} & -i\frac{x_1}{2} & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ -i\frac{x_N^*}{2} & -i\frac{y_N^*}{2} & 0 & 0 & \dots & 0 & 0 \\ i\frac{y_N}{2} & -i\frac{x_N}{2} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{2.216}$$

Among the $2N+2$ traces of matrix A, only three are functionally independent, leading to the first integrals

$$I_1 = H, \qquad I_2 = |x_0|^2, \qquad I_3 = \sum_{k=1}^{N} \left(|x_k|^2 - |y_k|^2\right). \tag{2.217}$$

Both A and B have a $2N$-dimensional kernel $\boldsymbol{\alpha}_0$, $\boldsymbol{\beta}_0$ (since there is a null submatrix of dimension $2N$). The intersection $\Psi = \boldsymbol{\alpha}_0 \cap \boldsymbol{\beta}_0$ has the form $\Psi = (0, 0, \psi_1, \varphi_1, \dots, \psi_N, \varphi_N)$ with

$$\psi_1 = |x_1|^2 + |y_1|^2, \qquad \varphi_1 = -\sum_{k=2}^{N} \left((x_1 y_k - x_k y_1)\psi_k + (x_1 x_k^* + y_1 y_k^*)\varphi_k\right). \tag{2.218}$$

Using the fact that ψ_k, φ_k are arbitrary constants for all k (since $\dot{\Psi} = 0$), we conclude that the quadratic combinations

$$J_k = |x_k|^2 + |y_k|^2, \qquad k = 1, \dots, N, \tag{2.219.a}$$

$$I_{k+2} = \sum_{j=1}^{k-1} (x_j x_k^* + y_j y_k^*)(x_j^* x_k + y_j^* y_k), \quad k = 2, \dots, N, \tag{2.219.b}$$

are additional first integrals. ■

2.12.4 Recycling integrable systems

The structure of the Lax pairs allows us to define new integrable systems that share the same first integrals. Hence, we can create new integrable systems from old ones. Let (A, B) be a regular Lax pair for $\delta_{\mathbf{f}}$ such that A has no non-zero constant entries and consider the *generalized Lax relation*

$$\dot{A} = [B, A] + \lambda A, \tag{2.220}$$

where λ is an arbitrary constant. This relation defines a new vector field $\delta_{\mathbf{g}}$ with first integral $I_k = \mathrm{Tr}(A^k)e^{-k\lambda t}$.

Example 2.38 Consider system (2.157) together with the Lax pair (2.158). The relation $\dot{A} = [B, A] + \lambda A$ defines a new vector field corresponding to the system

$$\dot{x} = (\mu_3 - \mu_2)yz + \lambda x, \tag{2.221.a}$$

$$\dot{y} = (\mu_1 - \mu_3)xz + \lambda y, \tag{2.221.b}$$

$$\dot{z} = (\mu_2 - \mu_1)xy + \lambda z. \tag{2.221.c}$$

This new dynamical system does not have a Lax pair. By contradiction, if it had a Lax pair, there would exist at least one time-independent first integral. However, this is impossible by Theorem 5.3. Nevertheless, it satisfies the generalized Lax relation and admits the first integral $I_1 = (x^2 + y^2 + z^2)e^{-2\lambda t}$. ■

More vector fields can be defined along the same lines by considering the generalized Lax relation

$$\dot{A} = [B, A] + \sum_{i=1}^{k} \lambda_i A^i. \tag{2.222}$$

The computation of the traces then leads to

$$\frac{d}{dt}\mathrm{Tr}(A) = \sum_{i=1}^{k} \lambda_i \mathrm{Tr}(A^i), \tag{2.223.a}$$

$$\frac{d}{dt}\mathrm{Tr}(A^2) = 2\sum_{i=1}^{k} \lambda_i \mathrm{Tr}(A^{i+1}), \tag{2.223.b}$$

$$\vdots,$$

$$\frac{d}{dt}\mathrm{Tr}(A^l) = l\sum_{i=1}^{k} \lambda_i \mathrm{Tr}(A^{i+l}). \tag{2.223.c}$$

Since the traces of A are functionally dependent, this system can be written as a system for $(n-1)$ independent quantities similar to system (2.99). This, in turn, defines first, second or third integrals.

2.12.5 More on Lax pairs

Other known Lax pairs and methods for finding Lax pairs include:

1. Systems with one and a half degrees of freedom (Deryabin, 1997);

2. Systems of N nonlinear coherent waves in multiple interactions (Menyuk *et al.*, 1982; Menyuk *et al.*, 1983; Romeiras, 1994);

3. Hamiltonians with two degrees of freedom and quartic potential. The idea here is to give a general form of Lax pairs compatible with potential that are separable (that is, for which there exists a set of coordinates that separate the equations in two separate integrable Hamiltonian systems);

4. The Winterniz system (Evans, 1990);

5. The Toda lattices (Flaschka, 1975; Flaschka, 1974; Bechlivanidis & van Moerbeke, 1987; Das & Okubo, 1989; Perelomov, 1990; Damianou, 1991; Ranada, 1995);

6. The Kovalevskaya top (Reyman & Semenov-Tyan-Shansky, 1987; Haine & Horozov, 1987; Reyman & Semenov-Tyan-Shansky, 1988; Bobenko *et al.*, 1989; Komarov & Kuznetsov, 1990; Audin & Silhol, 1993; Audin, 1996; Tsiganov, 1997; Marshall, 1998); and

7. The Goryachev-Chaplygin top (Bobenko & Kuznetsov, 1988; Audin, 1996) (Romeiras, 1995).

2.13 Exercises

2.1 Find the value of α such that the system

$$\dot{x} = x - y, \tag{2.224.a}$$
$$\dot{y} = -\alpha x + \alpha xy, \tag{2.224.b}$$

admits the time-dependent first integral $I = (y - 2x + x^2)e^{-2t}$. Compute the fixed points and show that a branch of the level set of this integral is a heteroclinic connection between the two fixed points. Find a closed form solution of this orbit.

2.2 Find a polynomial first integral for the system

$$\dot{x} = xz^2, \tag{2.225.a}$$
$$\dot{y} = z, \tag{2.225.b}$$
$$\dot{z} = -xyz - y^3 - 1. \tag{2.225.c}$$

2.3 Find a logarithmic first integral for the system

$$\dot{x} = -x + \frac{xy}{2} + xz + yz, \tag{2.226.a}$$
$$\dot{y} = xy - yz - y, \tag{2.226.b}$$
$$\dot{z} = z(2x + y). \tag{2.226.c}$$

Hint: the first integral has the form $I = g_0(x) + \log(g_1(x))$ where both g_0 and g_1 are linear in (x, y, z).

2.4 Let $I(\mathbf{x})$ be an analytic function of $\mathbf{x} \in \mathbb{R}^n$ and consider the *skew-gradient system*

$$\dot{\mathbf{x}} = A(\mathbf{x})\partial_{\mathbf{x}} I(\mathbf{x}), \tag{2.227}$$

where $A = A(\mathbf{x})$ is an analytic skew-symmetric matrix ($A^T = -A$). The system is a *Poisson system* if $A(\mathbf{x})$ satisfies the generalized *Jacobi-identity*

$$\sum_{l=1}^{n} (A_{il}\partial_{x_l} A_{jk} + A_{kl}\partial_{x_l} A_{ij} + A_{jl}\partial_{x_l} A_{ki}) = 0, \quad i, j, k = 1, \dots, n. \tag{2.228}$$

Show that $I(\mathbf{x})$ is a first integral. Show that the two previous systems are skew-gradient but not Poisson, and find the corresponding matrices A. This particular form has been used to device a numerical scheme preserving the first integral (Quispel & Capel, 1996; McLachlan & Quispel, 1998).

2.5 Use the Prelle-Singer algorithm to find an algebraic first integral for the system

$$\dot{x} = x^2 + 1, \tag{2.229.a}$$
$$\dot{y} = x^3 + x - xy, \tag{2.229.b}$$

2.6 The following model arises in cosmology as the Bianchi V model with a perfect fluid (Collins, 1996; Hewitt, 1991):

$$\dot{x} = a^2(1 - b^2)(1 - x^2) - y^2, \tag{2.230.a}$$
$$\dot{y} = -ab(c(1-x^2)-y^2) + 4(d-1)(2-d)xy, \tag{2.230.b}$$

where $1 < d < 2,\ b \geq 0,\ a > 0,\ a^2 = (3d - 2)(2 - d)$, and $c = 4(d - 1)(2 - d) + a^2(1 - b^2)$. Show that this system admits the second integral $ax^2 + y^2 = a$. Moreover, show that if $r = \sqrt{(3d - 2)/(2 - d)}$, then $[(3d - 2)x + y]\,[4(d - 1)x + y] = 1$ is another Darboux polynomial.

2.7 Find a planar vector field that admits a given polynomial function $J(x, y)$ as a Darboux polynomial. Show that if different branches of the level set $J(x, y) = 0$ intersect transversally at one point, then this point is a resonant fixed point, and some of the branches of the level set are part of a homoclinic or heteroclinic connection.

2.8 Consider the planar Lotka-Volterra homogeneous quadratic vector field

$$\dot{x} = x(ax + by), \quad (2.231.a)$$
$$\dot{y} = y(cx + dy). \quad (2.231.b)$$

Use Lemma 2.2 to find all the Darboux polynomials and compute the corresponding first integral. Find the conditions on (a, b, c, d) for such an integral to be logarithmic, rational or polynomial.

2.9 Add a diagonal linear term to the Lotka-Volterra system

$$\dot{x} = x(ax + by) + \lambda x, \quad (2.232.a)$$
$$\dot{y} = y(cx + dy) + \mu y, \quad (2.232.b)$$

and by using the result from the previous exercise, find conditions on λ and μ so that the system admits a first integral (Hint: Use the decomposition of the vector field into weight-homogeneous components).

2.10 Consider the general homogeneous quadratic vector field

$$\dot{x} = a_1x^2 + b_1xy + c_1y^2, \quad (2.233.a)$$
$$\dot{y} = a_2x^2 + b_2xy + c_2y^2. \quad (2.233.b)$$

How many Darboux polynomials can be found in general? Can you find restrictions on the parameters to find extra Darboux polynomials and eventually a first integral?

2.11 Consider the *Liénard system*

$$\dot{x} = y, \quad (2.234.a)$$
$$\dot{y} = -f_m(x)y - g_n(x), \quad (2.234.b)$$

where $f_m(x), g_n(x)$ are polynomials of degrees m and n respectively. One of the many interesting problems concerning the Liénard system is to prove the existence/non-existence of particular algebraic curves. For instance, Odani (1995) showed that if $n \leq m$ and $f_m g_n (f_m/g_n)' \neq 0$ then the Liénard system does not have an algebraic invariant curve (Żołądek, 1998a). Show that the particular system (Wilson, 1964)

$$f_2 = \mu(x^2 - 1), \quad g_5 = \frac{\mu^2 x^3}{16}(x^2 - 4) + x, \quad (2.235)$$

admits a *algebraic limit cycle* defined by the Darboux polynomial

$$J = \big(y + \mu x(x^2 - 4)\big)^2 + x^2 - 4. \quad (2.236)$$

2.12 Build a three-dimensional vector field that admits a quadric as a first integral and another quadric for third integral. Find the explicit algebraic invariant curves from the intersection of both quadrics.

2.13 Find the linear compatible system of (2.118) in the case $A = 1, B \neq 1$, and $C = -(B+1)^{-1}$. Compute the corresponding first integral.

2.14 Find the condition on the parameters of the Lorenz vector field (1.22) for which the system is compatible with the linear vector field defined by the matrix $L = \text{diag}(1, 2, 2)$. Compute the corresponding (rational) first integral.

2.15 Show that if M is a Jacobi multiplier, then M is a second integral and $\dot{M} = \alpha M$ with $\alpha = -\sum_{i=1}^{n} \partial_{x_i} f_i$.

2.16 Assume that an n-dimensional vector field admits two Jacobi multipliers M_1 and M_2. Show that $I = M_1/M_2$ is a first integral (this is a straightforward consequence of the previous problem).

2.17 Assume that $I_1, \ldots, I_{n-1}$ are the first integrals of a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and assume that I_{n-1} is the last integral found by Jacobi's method. The following identity gives the vector field as a function of the first integral (Cooke, 1984, p.208):

$$\mathbf{f} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ \partial_{x_1} I_1 & \partial_{x_2} I_1 & \cdots & \partial_{x_n} I_1 \\ \partial_{x_1} I_2 & \partial_{x_2} I_2 & \cdots & \partial_{x_n} I_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} I_{n-1} & \partial_{x_2} I_{n-1} & \cdots & \partial_{x_n} I_{n-1} \end{vmatrix}, \tag{2.237}$$

where $\mathbf{e}_i$ is the ith unit vector of the standard basis in $\mathbb{R}^n$. Show that the density of the invariant measure can be computed from the first integrals (Grammaticos *et al.*, 1990a) and can be expressed as

$$M = \begin{vmatrix} T_1 & T_2 & \cdots & T_n \\ \partial_{x_1} I_1 & \partial_{x_2} I_1 & \cdots & \partial_{x_n} I_1 \\ \partial_{x_1} I_2 & \partial_{x_2} I_2 & \cdots & \partial_{x_n} I_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} I_{n-1} & \partial_{x_2} I_{n-1} & \cdots & \partial_{x_n} I_{n-1} \end{vmatrix}, \tag{2.238}$$

where $\mathbf{T} = \frac{\mathbf{f}}{\mathbf{f}.\mathbf{f}}$ is a vector tangent to the flow.

2.18 **Liouville's theorem** Consider a two-degree-of-freedom Hamiltonian system $H = H(q_1, q_2, p_1, p_2)$ and assume that it admits an extra first integral $I = I(q_1, q_2, p_1, p_2)$. Use Jacobi's last multiplier theorem to show that $p_1 dq_1 + p_2 dq_2$ (where p_1, p_2 are evaluated on $H = h, I = c$) is the exact differential of a function $\Theta = \Theta(q_1, q_2, h, c)$. Moreover, show that $\partial_c \Theta =$constant and $\partial_h \Theta = t+$constant.

2.19 Compute the first integrals of the Toda lattice for $N = 4, 6, 8$. Show that in the general case, the matrix A given in Equation (2.161) provides N functionally independent first integral (Ranada, 1995).

2.20 Consider the n-dimensional system

$$\dot{x}_i = x_i(x_{i+1} - x_{i-1}), \qquad i = 1, \ldots, n, \tag{2.239}$$

where $x_{n+1} = x_1$ and $x_{-1} = x_n$. For $n = 3$, a Lax pair for the system is given by Equation (2.152). Show that for $n = 4$, matrix A for the Lax pair is

$$A = \begin{bmatrix} 0 & 1 & 0 & x_1 \\ x_2 & 0 & 1 & 0 \\ 0 & x_3 & 0 & 1 \\ 1 & 0 & x_4 & 0 \end{bmatrix}. \tag{2.240}$$

Find the corresponding matrix B. Find the Lax pairs for a general n, and show that the quantities $\mathrm{Tr}(A^k),\ k = 1, \ldots, n$ only define two functionally independent first integrals (Steeb & van Tonder, 1988b; Steeb & van Tonder, 1988a).

2.21 Consider the four-dimensional system for (x, y, u, v)

$$\dot{x} = y, \qquad \dot{y} = u + (a+b)x^2, \tag{2.241.a}$$

$$\dot{u} = v, \qquad \dot{v} = c - 1 - aux - 10a(a + \frac{b}{3})x^3. \tag{2.241.b}$$

Show that the following pair of differential operators form a Lax pair for the following given values of (a, b) where $\partial = \partial_t$:

• $(a, b) = (\frac{1}{2}, -3)$

$$A = \partial^2 + x, \tag{2.242}$$

$$B = 16\partial^5 + 40x\partial^3 + 60y\partial^2 + (50\dot{y} + 30x^2)\partial + 15\ddot{y} + 30xy, \tag{2.243}$$

•$(a, b) = (\frac{1}{2}, -\frac{1}{2})$

$$A = \partial^3 + x\partial, \tag{2.244}$$

$$B = 9\partial^5 + 15x\partial^3 + 15y\partial^2 + (50\dot{y} + 5x^2)\partial, \tag{2.245}$$

•$(a, b) = (\frac{1}{4}, -4)$

$$A = \partial^3 + 2x\partial^2 + y, \tag{2.246}$$

$$B = 9\partial^5 + 30x\partial^3 + 45y\partial^2 + (35\dot{y} + 20x^2)\partial + 10\ddot{y} + 20xy. \tag{2.247}$$

Find the corresponding two-dimensional Lax Pairs and compare them with the known results (Blaszak & Rauch-Wojciechowski, 1994).

Chapter 3:

Integrability: an analytic approach

C'est Painlevé qui, le premier, aborde l'étude globale des solutions dans le champ complexe, se rapprochant ainsi de Poincaré, qui le premier, avait fait une étude globale des courbes intégrales des équations différentielles dans le champ réel.[1]

R. Garnier, *in* (Painlevé, 1973)

In Chapter 2, the integrability of a system of differential equations was related to the existence of first integrals. First integrals express constraints on the dependent variables that can be used to simplify the system's dynamics. In this chapter, we study the behavior of the dependent variables as a function of the independent variable (say, the time). These functions are not explicitly known since it is generally not possible to solve systems of ordinary differential equations. Nevertheless, a local analysis can be performed around the singularities of the solutions without knowing their explicit form. Singularities of a solution are the locations in complex time at which the solution diverges. They are, in general, movable, that is, their location depends on the initial conditions. Therefore, we study the behavior of the solution seen as a function of a complex time. The *singularity analysis* is the local analysis of the solutions around their singularities in complex time.

Viewed as a function of complex time, the general solution of a system of differential equations can exhibit different types of behavior in their maximum domain of analytic continuation. The simplest such behavior is for the general solution to be single-valued. This property, known as the *Painlevé property*, imposes such strong conditions on the solutions that the system of differential equations which exhibits it may be considered "integrable". Unfortunately, there is no algorithmic procedure to decide if a system has the Painlevé property and only necessary conditions can be obtained. The procedures to obtain necessary conditions for the Painlevé property are known as *Painlevé tests* and impose that the solutions are meromorphic locally around the singularities. That is, the general local solution can be expressed as a Laurent series[2].

Most systems, however, do not enjoy the Painlevé property. Nevertheless, much critical information concerning the global behavior of the system can still be obtained from the local analysis around the singularities and the lack of meromorphicity can be used to prove the nonintegrability of the system. These aspects will be further analyzed in Chapters 5, 6 and 7.

This chapter introduces relevant definitions concerning the *singularity analysis* and the *Painlevé property* in the complex plane. A brief introduction to the singularities of linear differential equations

[1] Painlevé was the first to undertake a global study of the solution in the complex field much like Poincaré for the global study of integral curves of differential equations in the real field.

[2] A *Laurent series* around the point $t = t_0$ is a series of the form $\sum_{i=-\infty}^{\infty} a_i (t - t_0)^i$.

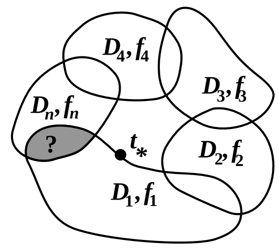

Figure 3.1: Analytic continuation around t_*.

is given and the Painlevé property for nonlinear differential equations is defined. The formal existence of the local series is demonstrated and the convergence of some partial series is discussed. Different Painlevé tests are derived in order of increasing complexity.

3.1 Singularities of functions

Let D be an open connected subset of the *Riemann sphere* $\Sigma = \mathbb{C} \cup \{\infty\}$. Consider a *function*, that is, a single-valued application and assume that it is analytic on a domain $f : D \to \Sigma$. When there exists another function $f' : D' \to \Sigma$ such that $D \cap D' \neq \emptyset$ and $f' = f$ on $D \cap D'$, then the function f can be uniquely continued on $D \cup D'$. This *analytic continuation* allows us to extends the function in a domain larger than the original domain.

At some points $t_* \in \Sigma$, the analytic continuation may not be possible (for instance $t_* = 0$ or ∞ for the log function). Let f be a function on D. If f can be analytically continued on $t_* \in \partial D$, then t_* is a *regular point*. That is, there exists $\epsilon > 0$ and an open disk B of center t_* and radius ϵ such that the function f can be extended in $D \cup B(t_*, \epsilon)$. If there is no such analytic continuation then t_* is a *singular point* or a *singularity* of f. In particular, if all points $t_* \in \partial D$ are singularities, then ∂D is a *natural boundary* of f on D.

Again, consider f on a domain D and assume that ∂D is not a natural boundary. Assume that all singularities on ∂D are *isolated singularities* (t_* is *isolated* if there exists $\epsilon > 0$ such that there are no singularities in the punctured open disk centered in t_* of radius ϵ). Let $t_* \in \partial D$ be an isolated singularities. Then, it is possible (see Figure 3.1) to build a finite sequence of functions f_i on D_i ($i = 1, \ldots, n$) around t_* and excluding other singularities of f_i such that $D_i \cap D_{i+1} \neq 0$, $D_n \cap D_1 \neq 0$ and $f_i = f_{i+1}$ on $D_i \cap D_{i+1}$. That is, f_{i+1} is a direct analytic continuation of f_i on D_{i+1}. What can we say about f_n on $D_n \cap D_1$? Either $f_1 = f_n$ or $f_1 \neq f_n$ on $D_n \cap D_1$. In the first case, f_n is a direct analytic continuation of f_1 on D_n and t_* is a *non-critical* singularity. In the second case, the repeated continuation of a single-valued function gives birth to a multi-valued application and the point t_* is referred to as a *critical singularity*.

The concept of continuation can be refined in order to consider the analytic (or meromorphic) continuation of a function along a path (a *path* is a continuous function $\gamma : [0,1] \to \Sigma$). The analytic continuation along a path from a to b is a finite sequence of direct analytic continuations on open discs D_i ($i = 1, \ldots, n$) with a subdivision $0 = s_0 < s_1 < \cdots < s_n = 1$ such that $\gamma([s_{i-1}, s_i]) \subseteq D_i \ \forall\, i$ and $a \in D_1, b \in D_n$. It is a standard matter to show that the analytic continuation along a path is unique (Jones & Singerman, 1987, p. 139) and only depends on the *homotopy class*[3] of γ. Therefore,

[3]Two paths γ_0, γ_1 between the points a and b in a topological space X are *homotopic* if there exists a continuous function $F(x, y) : [0,1] \times [0,1] \to X$ such that $F(x, 0) = \gamma_0(x), F(x, 1) = \gamma_1(x), F(0, y) = a, F(1, y) = b$. A *homotopy*

it is possible to define the continuation of a function in a domain where the equivalence between paths can be proved.

The two canonical examples for the occurrence of multi-valuedness are the log function (for each $z \in \mathbb{C}_0$, $\log(z)$ are the solutions of $e^w = z$) and the function $z^{1/q}$ ($q \in \mathbb{N},\ q \geq 2$ solutions of $w^q = z$). For these functions, the points 0 and ∞ are critical, that is, there is no meromorphic continuation around these points. However, in any region $D_J = \{z = re^{i\theta}; r > 0, \theta \in [\alpha, \beta]\}$ with $\alpha < \beta \leq \alpha + 2\pi$, the function f is single-valued and analytic. To define these functions, we may introduce *cuts* in the complex plane across which the function cannot be continued. However, this approach is not always satisfactory since the functions are not continuous on ∂D_J. Thus, instead of restricting the values of the function, the domain of definition is extended. The domain in which these functions are single-valued are known as the *Riemann surfaces*;, they are covering spaces of $\mathbb{C}_0$. For the logarithmic function, each *sheet* of the Riemann surface corresponds to a particular branch of $\log(z)$.

The *location* of the critical points is a crucial piece of information required to construct either the covering map for multi-valued applications (that is, many different functions on the same region) or the appropriate cuts in the complex plane. As we will see, the solutions of differential equations have singularities whose location may also depend on arbitrary constants.

3.2 Solutions of differential equations

A system of first order ODEs is a relation between the dependent variables and their derivatives. That is,

$$F_i(\mathbf{x}; \dot{\mathbf{x}}; t) = 0, \qquad i = 1, \dots, n. \tag{3.1}$$

For the purpose of this section we choose the functions F_i to be algebraic functions in $\{\mathbf{x}, \dot{\mathbf{x}}\} \subset \{\mathbb{C}^n, \mathbb{C}^n\}$, t denotes the independent variable and the dot $(\dot{\ })$ denotes the time derivative. A *solution* of this equation, in a path-connected domain $D \subset \mathbb{C}^n \times \mathbb{C}$ for given initial conditions $\{\mathbf{x}_0, t_0\} \in D$ is any function $x = x(t) \in D$ that satisfies the above equality and such that $\mathbf{x}(t_0) = \mathbf{x}_0$ If the Jacobian matrix $J(\mathbf{x}, \dot{\mathbf{x}}) = \partial_{\dot{\mathbf{x}}}\mathbf{F}$ is regular (*i.e.*, the Jacobian does not vanish) on a domain $D_{\mathbf{x}} \subset \mathbb{C}^n$, then using the implicit function theorem, the system of ODEs can be written locally as

$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}; t). \tag{3.2}$$

Let $\{\mathbf{x}_0, t_0\}$ be a point in a path-connected domain $D \subset D_{\mathbf{x}} \times \mathbb{C}$ where $\mathbf{G}$ is analytic. Then, Cauchy's theorem (Ince, 1956, p.71) guarantees the existence of a unique analytic solution $\mathbf{x} = \mathbf{x}(t)$ in a domain $D' \subset D$.

We can also consider the n initial conditions $\mathbf{x}_0$ as arbitrary constants of integration and define the *general solution* of an ODE in a domain $D' \subset D$ (where $\mathbf{G}$ is analytic) to be the solution with n arbitrary constants of integration. By contrast, a *particular solution* is any solution obtained by assigning values to at least one arbitrary constant. At the points where the Jacobian matrix is singular or $\mathbf{G}$ is not holomorphic[4], there may exist other types of solutions, the so-called *singular solutions.*

---

class is an equivalence class of homotopic paths.

[4]A function $f : D \subset \mathbb{C} \to \mathbb{C}$ is *holomorphic* if f is complex-differentiable for all points z in D. That is, $\lim_{h\to 0} \frac{f(z+h) - f(c) - f'(z)h}{h} = 0$. Note that a holomorphic function on D is also *analytic* (the function has local convergent power series expansions around z) and both designations are synonymous (Remmert, 1991, p.62). A function is *meromorphic* if it is holomorphic except on a set which contains the poles of the functions.

A singular solution of (3.1) satisfies $\det(J(\mathbf{x},\dot{\mathbf{x}})) = 0$. Singular solutions are not particular solutions since they cannot be obtained from the general solution by setting the arbitrary constants.

Example 3.1 A singular solution. Consider the equation

$$F(t,x,\dot{x}) = t\dot{x}^2 - 2x\dot{x} + 25t = 0. \tag{3.3}$$

The Jacobian matrix is given by $J(x,\dot{x}) = 2t\dot{x} - 2x$. The solution of the relation $J = 0$ together with Equation (3.3) gives the singular solutions $x = \pm 5t$. ■

Singular solutions can be of considerable interest since they allow us to compute envelopes for a family of solutions. Singular solutions can be obtained by eliminating the arbitrary constants appearing in the solutions $\mathbf{x} = \mathbf{x}(t;\mathbf{C}),\ \mathbf{C} \in \mathbb{C}^n$ together with the conditions $\det(J(\mathbf{x}(t;\mathbf{C}),\dot{\mathbf{x}}(t;\mathbf{C}))) = 0$.

Example 3.2 A circular singular solution. The differential equation

$$F(t,x,\dot{x}) = (t\dot{x} - x)^2 - \dot{x}^2 - 1 = 0, \tag{3.4}$$

is a *Clairaut equation*, that is, an equation of the form $f(t\dot{x} - x) = g(\dot{x})$ for which the general solution assumes the form $f(tC - x) = g(C)$ where C is an arbitrary constant (Zwillinger, 1989, p. 159). In this particular case, we have $f(x) = x^2$ and $g(x) = x^2 + 1$ and the general solution is

$$x = Ct \pm \sqrt{C^2 + 1}. \tag{3.5}$$

The general solution is holomorphic and therefore the equation has the Painlevé property (see Section 3.5). The singular solution is obtained by solving $J(x,\dot{x}) = 2t(t\dot{x} - x) - 2\dot{x} = 0$. That is, $\dot{x} = tx/(t^2 - 1)$, which upon substitution in (3.4) leads to the singular solution

$$x^2 + t^2 = 1. \tag{3.6}$$

This singular solution has an algebraic branch point at $t = \pm 1$. The singular solution is the envelope of the family of straight lines given by the general solution (see Figure 3.2). ■

Singular solutions and critical points will be important in the definition of the Painlevé property. The Painlevé property only addresses the meromorphicity of the general solution and singular solutions are not included in the definition. Singular solutions can exhibit, a priori arbitrary types of singularities without changing the main properties of the general solution.

3.3 Singularities of linear differential equations

In this section we review some basic results concerning linear systems of ODEs in the complex plane. The presentation is mostly expository and the proofs can be found in classical textbooks (Coddington & Levinson, 1955; Ince, 1956; Hille, 1969). By contrast to most classical textbooks, we present the results for systems of first order ODEs rather than the particular class of nth order ODEs for one variable.

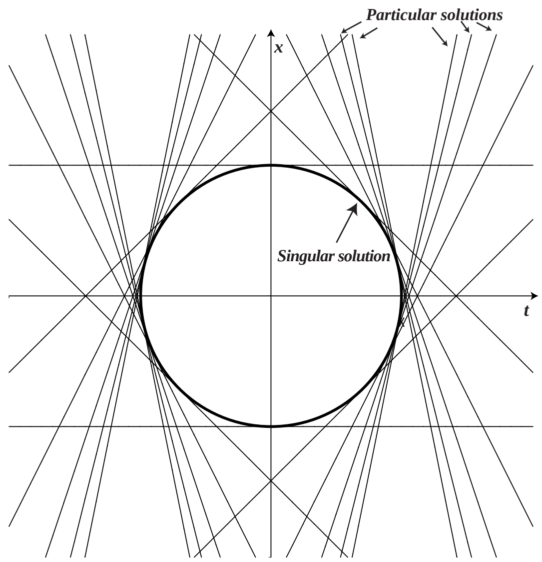

Figure 3.2: The singular solution and some particular solutions of Example 3.2 obtained from the general solution by taking values of C from -5 to +5.

3.3.1 Fundamental solutions

Consider the system of n linear ODEs

$$S(M;\mathbf{x}): \quad \dot{\mathbf{x}} = M(t)\mathbf{x}. \tag{3.7}$$

Suppose that $M(t)$ is holomorphic in the path-connected domain D (*i.e.*, every entry of M is holomorphic on D) and let $t_0 \in D$ be a given point. Consider the initial condition $\mathbf{x}_0 \in \mathbb{C}^n$. Then, there is a unique solution of $S(M;\mathbf{x})$ noted $\mathbf{x} = \mathbf{x}(t;t_0,\mathbf{x}_0)$ holomorphic in D such that $\mathbf{x}(t_0;t_0,\mathbf{x}_0) = \mathbf{x}_0$. Consider a matrix $X_0 \in \mathrm{GL}(n,\mathbb{C})$. Then, there is a matrix $X(t;t_0,X_0)$ which solves the matrix equation

$$\dot{X} = M(t)X, \tag{3.8}$$

and satisfies $X(t_0;t_0,X_0) = X_0$. Due to the linearity of the equation, if X_1 is an arbitrary constant matrix, then $X(t)X_1$ is also a solution. Therefore, we have

$$X(t;t_0,X_0) = X(t;t_0,\mathrm{I})X_0. \tag{3.9}$$

An important property of the matrix solution $X(t)$ is that $X(t;t_0,X_0)$ is invertible if and only if X_0 is invertible. That is, the *fundamental solution matrix* X is composed of linearly independent solutions vectors if and only if the initial vectors are linearly independent. An elegant way to prove this result is to derive the time evolution of the determinant of X (see Exercise 3). Let $\chi(t) = \det(X(t))$, then

$$\chi(t) = \chi(t_0)\exp\left\{\int_{t_0}^{t} \mathrm{tr}\big[M(s)\big]ds\right\}. \tag{3.10}$$

We conclude from this last expression that the fundamental solution matrix $X(t)$ is either regular everywhere or nowhere in D

3.3.2 Regular singular points

A fundamental solution can be analytically continued on any domain on which $M(t)$ of $\dot{X} = M(t)X$ can be continued. The problem is to investigate the singularity of $X(t)$ in the neighborhood of a singularity of $M(t)$. In general, the type of the singularities of $X(t)$ is not related to the type of the singularities of $M(t)$. The solution $X(t)$ can have a more complicated singularity or a simpler one. For instance, a simple pole of $M(t)$ may lead to a transcendental critical point, to a pole or even to holomorphic solution of $X(t)$. By contrast, a multiple pole of $M(t)$ may introduce an essential singularity in the solution.

Definition 3.1 Assume that in the neighborhood of the origin $M(t)$ has a simple pole, that is, $M(t) = \frac{1}{t}H(t)$ where $H(t)$ is holomorphic for $|t| < R \in \mathbb{R}$ and $H(0) \neq 0$. Then, the point $t = 0$ is a *regular singular point* of $S(M; \mathbf{x})$.

In the case of a regular singular point, the general solution assumes a particularly simple form (see for instance Coddington and Levinson (1955)).

Proposition 3.1 *If $t = 0$ is a regular singular point for the system $S(M, \mathbf{x})$, then there exist n linearly independent solutions of the form*[5]

$$\mathbf{x}_i = t^{\rho_i} \left(\mathbf{c}_{i0} + \mathbf{c}_{i1} \log t + \ldots + \mathbf{c}_{is_i} (\log t)^{s_i} \right), \qquad i = 1, \ldots, n, \tag{3.11}$$

with $\mathbf{c}_{ij} \in \mathbb{C}((t))^n$, and where ρ_i is an eigenvalue of $H(0)$ of algebraic multiplicity s_i.

If, moreover, the matrix H is constant we have the following result (see Exercise 4 for an explicit form of the solution).

Proposition 3.2 *If $t = 0$ is a regular singular point for the system $S(M, \mathbf{x})$ and $M = \frac{1}{t}H$ where H is a constant matrix, then its general solution is*

$$\mathbf{x} = t^H \mathbf{C}, \tag{3.12}$$

where t^H is the exponential matrix given by $t^H = e^{(\log t)H}$ and $\mathbf{C}$ is a vector of arbitrary constants.

If $M(t)$ has a multiple pole at $t = 0$ it is possible, in some instances, to find a linear transformation $\mathbf{x}' = L\mathbf{x}$ with $L \in \mathrm{M}_n(\mathbb{C}(t))$. This linear transformation maps the equation to a new equation with possibly a simple pole in $t = 0$ (Singer, 1990). If there is no such transformation, then $t = 0$ is an *irregular singular point* for which there exist n linearly independent solutions of the form

$$\mathbf{x}_i = e^{Q_i(t)} t^{\rho_i} \left(\mathbf{c}_{i0} + \mathbf{c}_{i1} \log t + \ldots + \mathbf{c}_{is_i} (\log t)^{s_i} \right), \tag{3.13}$$

where $\mathbf{c}_{ij} \in \mathbb{C}[[t^{1/q_i}]]^n$, $\rho_i \in \mathbb{C}; s_i, q_i \in \mathbb{N}$ and Q_i is a polynomial in t^{1/q_i}.

In general, the series in these solutions are not convergent and the solutions are referred to as *formal*. However, under some assumptions on the analyticity of $H(t) = M(t)t^q$ ($q \in \mathbb{N}$) near the origin, the formal solutions are asymptotic expansions of analytic solutions in a proper sector at the origin (Ramis & Martinet, 1989; Martinet & Ramis, 1991).

[5]$\mathbb{K}((t))$ and $\mathbb{K}[[t]]$ denote, respectively, the set of convergent and formal power series in t with coefficients in $\mathbb{K}$.

3.4 Singularities of nonlinear differential equations

3.4.1 Fixed and movable singularities

The singularities of linear differential equations are located at the singularities of their coefficients. Thus, they are *fixed* in the sense that they are known from the coefficients and t_* is a singularity of the solution only if it is a singularity of $M(t)$. The critical singularities can be therefore *uniformized*. That is, it is possible to introduce at each singularity the proper Riemann surface on which the solution is well defined. This property is lost for nonlinear differential equations where singularities can be either *fixed* or *movable*. A *movable singularity* is a singularity whose location in Σ depends on the initial conditions. The fixed singularities depend on the parameters of the system but not on the initial conditions and their locations are given by the singularities of the equation itself.

Example 3.3 A simple movable singularity. Consider the equation

$$\dot{x} = x^3, \tag{3.14}$$

with initial conditions $x(t_0) = x_0 > 0$. The general solution

$$x = \left[2(t_0 - t) + x_0^{-2}\right]^{-1/2}, \tag{3.15}$$

has a movable algebraic branch point at $t = t_0 - x_0^{-2}/2$. ■

Example 3.4 Different singularities. To further illustrate the variety of singularities that nonlinear ODEs may exhibit we consider the following examples (Ablowitz & Clarkson, 1991; Ince, 1956)

$$x\ddot{x} = \dot{x} - 1, \tag{3.16.a}$$
$$a x\ddot{x} = (a-1)\dot{x}^2 \qquad (a \in \mathbb{R} \setminus \mathbb{Q}), \tag{3.16.b}$$
$$\left[x\ddot{x} - \dot{x}^2\right]^2 = -4x\dot{x}^3, \tag{3.16.c}$$
$$(1 + x^2)\ddot{x} = (2x - 1)\dot{x}^2. \tag{3.16.d}$$

The general solutions of these equations are, respectively,

$$\begin{array}{ll} x(t) = (t-t_0)\log(t-t_0) + c(t-t_0) & \text{movable logarithmic branch point,} \\ x(t) = c(t-t_0)^a & \text{movable transcendental singularity,} \\ x(t) = c e^{\frac{1}{(t-t_0)}} & \text{movable essential singularity,} \\ x(t) = \tan\{\log[c(t-t_0)]\} & \text{movable essential singularity.} \end{array} \tag{3.17}$$

where c and t_0 are arbitrary constants. ■

3.5 The Painlevé property

For the solutions to be single-valued, all singularities should be non-critical. However, since their position is known, fixed singularities can be uniformized, either by introducing cuts in the complex plane across which the solution cannot be continued or, equivalently, by introducing the proper covering surfaces. Therefore, the only types of singularities that can prevent the general solution from being single-valued are the movable critical singularities. This remark leads to the definition of the Painlevé property.

Definition 3.2 A system of ODEs $\mathbf{F}(t; \mathbf{x}, \dot{\mathbf{x}}) = 0$ enjoys the *Painlevé property* if the general solution $\mathbf{x} = \mathbf{x}(t; C_1, \ldots, C_n)$ has no movable critical singularity.

Note that the Painlevé property only depends on the general solution and not on the singular solutions. For example, the equation in Example 3.2 has a holomorphic general solution whereas the singular solution has a fixed algebraic singularity (see also Exercise 1 where an equation with an holomorphic general solution and a singular solution with logarithmic movable branch point is given).

As defined above, the Painlevé property does not allow for movable critical singularities. Assume that for given initial conditions $\mathbf{C} \in \mathbb{C}^n$, the general solution is holomorphic in a domain $D \in \mathbb{C}^n$, and consider the maximal domain of continuation D_{max} of $\mathbf{x}(t, \mathbf{C})$, $D \subseteq D_{\text{max}} \subset \mathbb{C}^n$. Let t_* be a singularity (isolated or not) of $\mathbf{x}(t, \mathbf{C})$ on ∂D_{max}, it is *fixed* if it is a singularity of $\mathbf{x}(t, \mathbf{C})$ for all $\mathbf{C}$ and *movable* otherwise. It is *critical* if there exists a closed path $\gamma \in D_{\text{max}}$ around t_* such that the only singularity inside γ is t_* and the continuation of $\mathbf{x}(t, \mathbf{C})$ is not analytic. In particular, if the solution has a closed *natural boundary* (*i.e.*, a curve $\gamma \in \mathbb{C}$ such that all points $t_* \in \gamma$ are singularities of the general solution), the equation still may have the Painlevé property as long as the solutions inside and outside the curve are both single-valued (for instance, the Chazy equation (3.173) has the Painlevé property despite the occurrence of a movable boundary).

The Painlevé property is a global property. *A priori*, it requires an explicit form of the general solution for all values of the independent variables and the arbitrary constants. Generally, it is not possible to obtain the general solution, and the Painlevé property has to be proven on a case by case basis.

3.5.1 Historical digression I

To understand the importance of the Painlevé property, we revisit some of the classical works on the integration of differential equations. In early seventeenth-century, the Scottish Lord Napier was interested in one of the first differential equations ever studied. It was proposed by Galileo as a model for uniform motion and reads (in modern notation)

$$\dot{x} = x \tag{3.18}$$

Whereas Galileo understood that the motion could not be realized in the physical setting he was considering (free motion in a gravity field), Napier studied the mathematical property of this equation. He found some remarkable properties for its solution and its inverse which lead him to introduce two new functions. After twenty years of laborious work, it eventually led to the concept of exponentials and logarithms. In 1614, he published *Mirifici Logarithmorum Canonis Descriptio (A Description of the Wonderful Law of Logarithms)*, the first ever table of logarithms (Gindkin, 1988). It was the first instance where the solution of a differential equation was used to define a new mathematical function.

In late nineteenth-century, following the work of L. Fuchs on linear and nonlinear ODEs, Paul Painlevé carried out an extensive and innovative work on the classification of first and second order differential equations. The main idea behind this work was to define new functions as solutions of linear or nonlinear ODEs and was a natural extension of the celebrated works of Abel and Jacobi on the equation

$$(\dot{x})^2 = (1 - x^2)(1 - k^2 x^2), \tag{3.19}$$

the solution of which is single-valued in the complex plane. However, it cannot be expressed in terms of elementary functions and new functions, the well-known *elliptic sine functions* $x = \mathrm{sn}(\mathrm{t} - \mathrm{t}_0; \mathrm{k})$

have to be introduced (Lang, 1987). Various works on elliptic, Fuchsian and other special functions are related by a fundamental classification problem formulated by Painlevé (Painlevé, 1973, p. 135):

> *Déterminer toutes les équations algébriques du premier ordre, puis du second ordre, puis du troisième ordre, etc., dont l'intégrale générale est uniforme.*

A few comments are in order: the *intégrale générale* means *the general solution* and *uniforme* is best translated by *single-valued*. In translation, Painlevé's problem reads:

> *[The problem is] to determine all algebraic differential equations of first order, second order, third order, etc., such that the general solution is single-valued.*

At the time of Painlevé, the solution to the first order problem had already been studied by L. Fuchs, who considered the differential equations

$$\dot{x} = \frac{P_1(x;t)}{P_2(x;t)}, \tag{3.20}$$

where P_1, P_2 are polynomials in $x \in \mathbb{C}$ and analytic in t. Fuchs showed that the only equation whose solution's singularities are critical fixed points depending solely on the singularities of the coefficients is the generalized Riccati equation (Ince, 1956, p. 293)

$$\dot{x} = a(t)x^2 + b(t)x + c(t). \tag{3.21}$$

Subsequently, Painlevé considered the algebraic first order equations with movable poles or algebraic branch points of the form

$$G(\dot{x}, x; t) = 0, \tag{3.22}$$

where G is polynomial in $x, \dot{x}$ and analytic in t. The next step was the classification of second-order differential equations of the form

$$\ddot{x} = F(\dot{x}, x; t), \tag{3.23}$$

where F is polynomial in $\dot{x}$, algebraic in x and analytic in t. The problem is to find all equations (3.23) whose general solution has no movable critical point. That is, all the equations enjoying the Painlevé property (for more on the history of the Painlevé equations see the review article by Conte (1999)). While first order equations may have only algebraic branch points as movable critical points, second order equations also exhibit logarithmic branch points and essential singularities. To study these equations, Painlevé developed a new original method, the α-method (see Section 3.5.2). The α-method gives conditions for the absence of logarithmic branch points. Most of the equations (3.23) found by this method can be readily integrated by special functions or quadratures. However, Painlevé found three equations which could not be reduced to known functions and carefully studied them to prove first the absence of essential singularities and second their irreducibility to other functions. The simplest such equations is the so-called PI-equation (for Painlevé one)

$$\ddot{x} = 6x^2 + t. \tag{3.24}$$

Painlevé realized that his classification was incomplete. Eventually, one of his students, Bertrand Gambier, completed his work and found three new equations (one of which, the so-called PVI, was first discovered by Fuchs in a different context). As a result, Painlevé and colleagues found that within the class (3.23) only 53 canonical equations enjoy the Painlevé property. Among them, 47 can be integrated in terms of known functions (Weierstrass and Jacobi elliptic functions). The 6 remaining ones are the *Painlevé equations* (PI $\to$ PVI), whose solutions require the introduction of new transcendents, the *Painlevé transcendents*[6] (see Table 3.1). The explicit goal of Painlevé's work is to define new functions and classify classes of equations (first, second, third order,... ODEs). That is, to find all equations which share a common property, namely, the Painlevé property. The classification of third order ODEs has not been completed yet despite some extensive work (Bureau, 1987) (see also Conte (1999) for an exhaustive discussion).

$$\ddot{x} = 6x^2 + \lambda t$$

$$\ddot{x} = 2x^3 + tx + \mu$$

$$t x\ddot{x} = t\dot{x}^2 - x\dot{x} + at + bx + cx^3 + dtx^4$$

$$x\ddot{x} = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}a^2 + 2(t^2 - b)x^2 + 4tx^3 + \tfrac{3}{2}x^4$$

$$\begin{aligned} t^2(x - x^2)\ddot{x} = {} & \tfrac{1}{2}t^2(1 - 3x)\dot{x}^2 - tx(1 - x)\dot{x} + ax^2(1 - x)^3 \\ & + b(1 - x)^3 + ctx(1 - x) + dt^2x^2(1 + x) \end{aligned}$$

$$\begin{aligned} x(1 - x)(t - x)\ddot{x} = {} & \tfrac{1}{2}\left[t - 2(t + 1)x + 3x^2\right]\dot{x}^2 + \tfrac{x(1-x)}{t(1-t)}\left[t^2 + (1 - 2t)x\right]\dot{x} \\ & + \tfrac{1}{2t^2(1-t)^2}\left[ax^2(1 - x)^2(t - x)^2 - bt(1 - x)^2(t - x)^2 \right. \\ & \left. - c(1 - t)x^2(t - x)^2 - dt(1 - t)x^2(1 - x)^2\right] \end{aligned}$$

Table 3.1: The six Painlevé equations: λ, μ, a, b, c, d are arbitrary parameters.

3.5.2 Painlevé's α-method

The innovative idea of Painlevé was to generalize the notion of small parameters used in perturbation theory by introducing an *artificial* parameter, α, in the equation. If the equation is single-valued for all $\alpha \neq 0$, then it is also single-valued for $\alpha = 0$. The main difficulty is to choose a good parameter α such that the equation for $\alpha = 0$ can be immediately integrated. This can be achieved, for instance, by using a scaling symmetry. Consider the first Painlevé equation (3.24) and introduce the scaling symmetry $X = \alpha^2 x$ and $T = \alpha^{-1}t$. In the limit, $\alpha \to 0$, PI reduces to

$$\frac{d^2X}{dT^2} = 6X^2. \tag{3.25}$$

This equation is solved in terms of the Weierstrass function, $X(T) = \wp(T - c_1; 0, c_2)$ where c_1, c_2 are integration constants (Lang, 1987). The exact solution of this equation with $\alpha = 0$ provides a starting

[6]The solutions of the Painlevé equations are not the Painlevé transcendents but combinations of them.

point in a perturbative expansion in powers of α. More generally, consider a system of equations of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; t), \tag{3.26}$$

where $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{f}$ is polynomial in $\mathbf{x}$ and analytic in t. We introduce the change of variables

$$t = \alpha^{p_1} T + \alpha^{p_2} a, \tag{3.27.a}$$

$$\mathbf{x} = \alpha^{q_1} \mathbf{X} + \alpha^{q_2} \mathbf{b}, \tag{3.27.b}$$

with $p_1,\ p_2,\ q_1,\ q_2 \in \mathbb{Z}$ and a, $\mathbf{b}$ are constants. After a suitable transformation, that is, for a convenient choice of p_1, p_2, q_1, q_2, we obtain the system

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}; T; \alpha), \tag{3.28}$$

which can be exactly integrated in the limit $\alpha = 0$. Now, expand $\mathbf{X}$ in series of α:

$$\mathbf{X}(T) = \sum_{i=0}^{\infty} \mathbf{X}^{(i)}(T)\alpha^i, \tag{3.29}$$

and prove the single-valuedness of each $\mathbf{X}^{(i)}$. The crucial result for this analysis is a theorem due to Painlevé (see Kruskal and Clarkson (1992) for a modern formulation and Golubev (1953) for a complete proof).

Theorem 3.1 (Painlevé's α-lemma) *Let $\mathbf{F}(\mathbf{X}; T; \alpha)$ be analytic in a path-connected domain $(\mathbf{X}, T, \alpha) \in (D_{\mathbf{X}} \subset \mathbb{C}^n, D_T \subset \mathbb{C}, D_\alpha \subset \mathbb{C})$. For given initial conditions $(\mathbf{X}(T_0), T_0) \in D_{\mathbf{X}} \times D_T$ and $\epsilon \in \mathbb{R}_0^+$ sufficiently small, the general solution of (3.28) is single-valued for all α such that $|\alpha| < \epsilon$ if and only if $\mathbf{X}^{(i)}(T)$ is single-valued for all i and $\mathbf{X}^{(i)}$ defined in (3.29).*

This fundamental result of Painlevé was used to derive necessary conditions for the Painlevé property. The first order equation is nonlinear (see for instance Equation (3.25)) while the higher order equations for the functions $\mathbf{X}^{(i)}$ are linear. The classical results of Fuchs on regular singular points can then be applied to find the occurrence of critical movable branch points. The drawback of this method is that the infinite hierarchy of linear differential equations for the functions $\mathbf{X}^{(i)}$ cannot be solved in general. However, a local analysis around the movable singularities allows us to find necessary conditions for the absence of branch points. The next step is to prove the absence of a critical essential singularity. Generally, this cannot be performed algorithmically.

Example 3.5 An essential singularity. To illustrate this difficulty, consider the equation (Ince, 1956, p. 344)

$$\ddot{x} = \left[\frac{x[2k^2x^2 - (1+k^2)]}{(1-x^2)(1-k^2x^2)} - \frac{1}{\lambda\sqrt{(1-x^2)(1-k^2x^2)}} \right] \dot{x}^2. \tag{3.30}$$

The general solution of this equation has no algebraic branch points. Moreover, it can be proved that the singularities of any solution, $x = x(t)$, which has a limit when $t \to t_*$ are either analytic or non-critical. However, the solution is not meromorphic in the complex plane since the general solution

$$x = \operatorname{sn}\{\lambda \log(C_1 t - C_2)\} \pmod{k}, \tag{3.31}$$

where C_1, C_2 are arbitrary constants, does not have the Painlevé property. The point $t_* = C_2/C_1$ is an essential critical singularity of the solution and the limit as $t \to t_*$ along any path is not defined. ■

This example demonstrates that the conditions provided by Painlevé's α-method are not sufficient. However, for the Painlevé equations, Painlevé and co-workers succeeded in showing the absence of essential singularities from a detailed analysis of the behavior of the solution around their singularities.

3.5.3 The isomonodromy deformation problem

The enumeration of applications and studies on the Painlevé equations is endless. From their discovery many new and interesting features have been discovered. One of the most important results is the relationship with the Riemann-Hilbert problem. Richard Fuchs and Ludwig Schlesinger, for the case of second order ODEs and for the case of systems of ODEs, respectively, showed that PVI was a solution to the isomonodromy deformation problem. Consider the system

$$\dot{\mathbf{x}} = \sum_{j=1}^{n} \frac{A_j}{t - t_j}\mathbf{x}, \tag{3.32}$$

where $\mathbf{x} \in \mathbb{C}^m((t))$ and $A_j \in \mathrm{M}_m(\mathbb{C})$. The fundamental solution X of this linear system is, in general, multi-valued around the singularity t_j. That is, $X(t_j + (t - t_j)e^{2\pi i})$ is not equal to $X(t)$, however, they are related by the so-called *monodromy matrix* $M_j \in \mathrm{M}_m(\mathbb{C})$, so that

$$X(t_j + (t - t_j)e^{2\pi i}) = M_j X(t). \tag{3.33}$$

The isomonodromy deformation problem is to *find A_j as a function of t_j such that the monodromy matrices M_j remain fixed.* The solution given by Schlesinger is (1912)

$$\frac{\partial A_j}{\partial t_r} = \frac{[A_j, A_r]}{t_j - t_r}, \; j \neq r, \qquad \sum_{j=1}^{n} \frac{\partial A_j}{\partial t_j} = 0. \tag{3.34}$$

For $m = 2$, three points 0, 1 and ∞ can be fixed, the fourth depends on a parameter s:

$$\ddot{x} = F(x, s)x, \tag{3.35}$$

where $F(x, s)$ is rational with singularities located on $0, 1, \infty$ and s. In order for the monodromy matrices to be independent of s, the variable t as a function of s must satisfy the PVI equation.

3.5.4 Applications

It seems that the Painlevé equations are ubiquitous and can be found in every field of mathematical physics. In hydrodynamics and plasma theory, the Painlevé equations are usually obtained as reduced ODEs of some PDEs describing the evolution of the flow, such as the KdV equation (see next section), or convective flows with viscous dissipation (Holmes & Spence, 1984). In nonlinear optics, the nonlinear Schrödinger equation plays a central role in the description of wave propagation in media, the reduction of which naturally leads to the Painlevé equations (Giannini & Joseph, 1989). They have also been found in general relativity (MacCallum, 1983) and quantum field theory (Creamer *et al.*, 1981).

In equilibrium statistical physics, solvable models were successfully explored and many new models have been found (Baxter, 1982). Interestingly, these solvable models seem to be closely related to the Painlevé equations. The spin-spin correlation of the rectangular two-dimensional Ising model has been connected to the third Painlevé transcendent (Barouch *et al.*, 1973; Wu *et al.*, 1976; Kadanoff & Kohmoto, 1980) and the fifth Painlevé transcendent to the n-particle reduced density matrix of the one-dimensional impenetrable Bose gas (Jimbo *et al.*, 1981; McCoy & Tang, 1986; Creamer *et al.*, 1986). Is there a deeper connection between solvable models in statistical mechanics and integrability theories? Would it be possible to define a Painlevé-like test for statistical models and derive from it relations such as the Yang-Baxter relations? To date, this problem has not been been solved (Flaschka *et al.*, 1991, p. 78).

3.6 Painlevé equations and integrable PDEs

For more than half a century after its development, the Painlevé theory for differential equations was considered an interesting, if not old fashioned, masterpiece of the theory of special functions and little attention was paid to it until striking connections with soliton theories were made in the early 1980s. To understand the central role played by singularity analysis, we briefly discuss the theory of solitons and its intimate connection with the theory of Painlevé's equations.

3.6.1 The theory of solitons and the Inverse Scattering Transform

In 1965, Zabusky and Kruskal discovered that some solutions of the Korteweg-de Vries equation (KdV) (Korteweg & de Vries, 1895);

$$u_t + 6uu_x + u_{xxx} = 0, \tag{3.36}$$

interact elastically with each other. This particle-like behavior indicated the possibility of a regular behavior for the solution and gave rise to a new integrability theory for PDEs, the theory of *solitons*. Subsequently, it was found by Miura *et al.* (Miura, 1968; Miura *et al.*, 1968) that the KdV equation is "integrable" in the sense that it has an infinite number of conservation laws, the first of which read

$$\int_{-\infty}^{+\infty} u dx = c_1, \tag{3.37.a}$$

$$\int_{-\infty}^{+\infty} u^2 dx = c_2, \tag{3.37.b}$$

$$\int_{-\infty}^{+\infty} (u^3 + \frac{1}{2}u_x^2) dx = c_3, \tag{3.37.c}$$

$$\vdots$$

Moreover, their work showed an intimate relationship between the KdV equation, the modified KdV equation, and the Schrödinger equation. Based on these observations, a new method for computing the exact solution of the KdV equation was proposed, the so-called *inverse scattering transform* (IST) (Gardner *et al.*, 1967; Zakharov & Shabat, 1972; Ablowitz *et al.*, 1973; Gardner *et al.*, 1974). For a detailed study of the IST and applications, see the books and reviews by Ablowitz and Segur (1981), Newell (1983; 1985), and Ablowitz and Clarkson (1991). Many new integrable systems were found and proved to be integrable by using the IST, such as

$$\begin{array}{ll} u_t - 6u^2u_x + u_{xxx} = 0 & \text{modified KdV,} \\ u_{tt} - u_{xx} + \sin u = 0 & \text{Sine} - \text{Gordon,} \\ iu_t = u_{xx} + 2u|u|^2 & \text{Nonlinear Schrödinger,} \\ u_{tt} = u_{xx} - 6(u^2)_{xxx} + u_{xxxx} & \text{Boussinesq.} \end{array} \tag{3.38}$$

3.6.2 The Ablowitz-Ramani-Segur conjecture

The integrability of these equations was not connected to their singularity structure until 1977 when Ablowitz and Segur noticed that some similarity reductions of integrable PDEs transform them into ODEs whose solutions can be expressed in terms of the Painlevé transcendents. For example, the KdV equation has a similarity reduction given by

$$u(x,t) = w(z) - \lambda t, \qquad z = x + 3\lambda t^2, \tag{3.39}$$

where λ is a constant and $w(z)$ is a solution of PI, that is,

$$w'' = 6w^2 + z. \tag{3.40}$$

Another possible reduction is

$$u(x,t) = -(3t)^{-2/3} w(z), \qquad z = \frac{x}{(3t)^{1/3}}. \tag{3.41}$$

It transforms the KdV equation into

$$w''' + 6(w - z)w' - 2w = 0, \tag{3.42}$$

which again is related to the solution of PII by the Miura transformation relating the KdV equation to the modified KdV equation.

There exist other similarity reductions for the integrable equations (3.38). They all share the property that the reduced ODEs are related to the 53 canonical equations classified by Painlevé and co-workers. To date, there are general methods to compute the similarity reductions of PDEs such as the Lie-group theoretical methods (Olver, 1993; Bluman & Cole, 1974) or the direct method of Clarkson and Kruskal (Clarkson & Kruskal, 1989; Clarkson, 1992; Clarkson & Mansfield, 1993; Nucci & Clarkson, 1992).

These observations were the crucial step to connect the integrability of PDEs with singularities in the complex plane and gave rise to the so-called Painlevé conjecture (proposed by Ablowitz, Ramani and Segur (1980a) and Hasting, McLeod (1980)).

> **Conjecture:** *Every ordinary differential equation, obtained by a similarity reduction of an integrable PDE (in the sense of the IST), has the Painlevé property (up to a change of dependent variables).*

This conjecture was proved for particular cases by Ablowitz *et al.* (1980b), and McLeod and Olver (1983). If true, the Painlevé conjecture could be used to detect new integrable systems, or at least prove the nonintegrability of some PDEs. It is sufficient to find a similarity reduction whose reduced ODE is not of Painlevé type to prove the nonintegrability of the corresponding PDE. However, even if the conjecture could be rigorously proved, there are two main difficulties with this procedure. First, in order to prove integrability all possible reductions have to be computed. Unfortunately, there is no decision procedure to find all similarity reductions. Worst, there are systems which do not have any similarity reductions and thus cannot be tested at all. Second, the reduced ODE may have the Painlevé property only after a nontrivial change of variables. Moreover, the Painlevé conjecture, as an effective tool for finding integrable PDEs, only provides necessary conditions for a system to be integrable and no information on the actual solution of the PDE. To circumnavigate these difficulties, a new method applicable directly to PDEs and similar to the Kovalevskaya approach for ODEs was proposed (discussed in detail in Section 3.8), the so-called PDE Painlevé test.

3.7 The PDE Painlevé test

Consider an evolution equation of the form

$$u_t = F(u, u_x, u_{xx}, \dots), \tag{3.43}$$

where F is polynomial in u and its derivatives with respect to $x \in \mathbb{C}$. The Painlevé conjecture places some constraints on the solution, mainly, it suggests that the solution should be meromorphic. To test this conjecture, Weiss, Tabor and Carnevale (1983) proposed a new method based on an new type of expansion, the so-called *WTC-expansion* (or *singular manifold expansion*). As explicitly shown in the following sections, in order to test the Painlevé property for ODEs, a simple test is to find the Laurent expansion for the solution. Now, if we translate this idea into the PDE setting, a Painlevé PDE test should consist of finding a singular expansion around the set of singularities of $u(x,t)$. This set could be written

$$\Phi(x,t) = 0, \tag{3.44}$$

where Φ is analytic in the neighborhood of the locus defined by the former relation. We require that the solutions can be expanded around the singularity manifold Φ. That is, all solutions in the neighborhood of $\Phi = 0$ can be formally expanded in the form

$$u(x,t) = (\Phi(x,t))^p \sum_{i=0}^{\infty} a_i(x,t)\,(\Phi(x,t))^i\,, \tag{3.45}$$

where p is an integer and the functions $a_i(x,t)$ are analytic around $\Phi = 0$. If we insert this ansatz into equation (3.43) and equate the coefficients a_i to each order in Φ, a recursion relation can be found for the coefficients a_i and their derivatives. These conditions are differential equations relating a_i to a_j $(j < i)$ and their derivatives. Some of the functions a_i are arbitrary functions which are prescribed by the Cauchy-Kovalevskaya theorem. The positions in the series where they first appear are called, by analogy with the Painlevé tests, the *resonances*. If all possible series expansions are of the form (3.45), then the equation passes the Painlevé PDE test. This test is much easier to perform than the IST. The new conjecture associated with the test is as follows.

> **Conjecture (WTC):** *If a PDE satisfies the Painlevé PDE test, then it is integrable (in the sense of the IST).*

While the results of Hasting and McLeod (1983) suggest that this conjecture is likely to be true, there is no proof or even partial proof of this statement. However, a considerable amount of evidences tends to prove that the Painlevé PDE test is certainly related to integrability. First, there has been numerous applications of the Painlevé PDE test to find new integrable systems (Weiss, 1983; Weiss, 1984; Weiss, 1985a; Weiss, 1985b; Weiss, 1989; Baumann, 1992; Gibbon & Tabor, 1985; Konopelchenko & Strampp, 1991; Zhu, 1993). Second, it has been shown that the Painlevé PDE test is connected to other integrability theories and may be used to compute crucial information on complete integrability such as Lax pairs (Tabor & Gibbon, 1986; Baby, 1987; Gibbon *et al.*, 1988; Musette & Conte, 1991), Bäcklund transformations (Weiss, 1989; Hu, 1993), n-soliton solutions, Hirota bilinear property (Grammaticos *et al.*, 1990b; Gibbon *et al.*, 1985; Hirota, 1986; Clarkson, 1986; Tabor & Gibbon, 1986) and special solutions (Chudnovsky *et al.*, 1983; Weiss, 1985a; Newell *et al.*, 1987; Kudryashov, 1988; Cariello & Tabor, 1989; Kudryashov, 1990; Cariello & Tabor, 1991; Kudryashov, 1992).

To discuss some of these features, we present an application of the PDE Painlevé test to the canonical example, the KdV equation, for which the soliton theory was designed initially.

Example 3.6 The Korteweg-de Vries equation. Before checking the Painlevé property for the Korteweg-de Vries (KdV) equation, we simplify expansion (3.45). The expansion variable $\Phi(x,t)$ is a function of two variables. Since $\Phi_x(x,t) \neq 0$ in a neighborhood of $\Phi(x,t)=0$, we can apply the implicit function theorem and replace $\Phi(x,t)$ in the neighborhood of the singular manifold by $\Phi(x,t)=x-\Psi(t)$ and $a_i = a_i(t)$ (this ansatz is sometimes refer to as the *reduced ansatz* or the *Kruskal ansatz—textit*). The algebraic manipulations are now greatly simplified since $\Phi_x(x,t)=1$ and all higher derivatives vanish identically. Other choices of $\Phi(x,t)$ are possible and some "gauges" further simplify the manipulations mainly in the computation of Lax pairs and Bäcklund transformations (Conte, 1988; Conte, 1991; Musette & Conte, 1991; Conte, 1994).

The first step is to find the leading behavior of the solution, that is, the exponent p and the first coefficient $a_0(t)$. It is an exact solution of the equation, that is, $u=-2\Phi(x,t)^{-2}$. Moreover, a simple computation shows that the arbitrary functions of the general solution are given by a_4 and a_6. Therefore, we have to check that the arbitrariness of these coefficients does not prevent the existence of a series (3.45). We expand the solution up to Φ^4, that is,

$$u(x,t) = \Phi^{-2}\left(-2+\sum_{i=1}^{6} a_i\Phi^i + O(\Phi^7)\right), \tag{3.46}$$

and we equate to each power in Φ the coefficients. After some algebraic manipulations (using the property $\Phi_x = 1,\ a_x = 0$), we obtain

$$a_1 = 0, \tag{3.47.a}$$
$$a_2 = -\Phi_t/6, \tag{3.47.b}$$
$$a_3 = 0, \tag{3.47.c}$$
$$0.a_4 = 0, \tag{3.47.d}$$
$$a_5 = -\Phi_{tt}/36, \tag{3.47.e}$$
$$0.a_6 = -2a_4\Phi_t - 12a_2a_4. \tag{3.47.f}$$

We find that a_4 and a_6 are arbitrary since $-2a_4\Phi_t - 12a_2a_4 = 0$. Therefore, the coefficients $a_i (i>6)$ can be obtained recursively which proves the formal existence of the series

$$u(x,t) = \sum_{i=0}^{\infty} a_i\Phi^{i-2}. \tag{3.48}$$

■

3.7.1 Integrability of ODEs

The Painlevé conjecture for PDEs gave rise to numerous works and it was soon realized that a similar connection between integrability and the Painlevé property could be established for ODEs. Ablowitz, Ramani and Segur (1980a) designed an algorithmic procedure to find necessary conditions for the Painlevé property in ODEs, the so-called *ARS algorithm*. As a consequence, many studies have been devoted to the relationship between integrability of Hamiltonian systems and non-Hamiltonian systems using the ARS algorithm (Bountis *et al.*, 1982; Bountis *et al.*, 1983; Dorizzi *et al.*, 1983; Bountis *et al.*, 1984; Ramani *et al.*, 1984; Grammaticos *et al.*, 1985; Dorizzi *et al.*, 1986; Hlavaty, 1988; Steeb & Euler, 1988; Goriely, 1992). However, these studies were all based on two implicit assumptions. First, the ARS algorithm does provide necessary conditions for the Painlevé property. Second, the Painlevé property is a necessary condition for "integrability". However, these two assumptions are not well-defined. In clarifying the notions of Painlevé property and integrability, some precise connections between these two global properties can be established.

The solutions of Liouville integrable systems (see Sections 2.10.2 for a discussion and Section 6.2.1 for a definition) are generally not single-valued. The Arnold-Liouville procedure requires n first integrals but the remaining $(n-1)$ first integrals (the angles) obtained by the Hamilton-Jacobi procedure are generally multi-valued. Therefore, if we want to establish connections between a geometric view of integrability and the Painlevé property, it is necessary to introduce more stringent definitions of integrability. Two new definitions of integrability have been proposed, one by Adler and van Moerbeke (Adler & van Moerbeke, 1987; Adler & van Moerbeke, 1989b), and the other by Ercolani and Siggia (Ercolani & Siggia, 1986; Ercolani & Siggia, 1989; Ercolani & Siggia, 1991).

Adler and van Moerbeke considered polynomial Hamiltonian systems.[7] They define algebraic complete integrability for these systems as the existence of sufficiently many polynomial constants of motion in involution whose level sets on $\mathbb{C}^n$ are related to Abelian variety (Adler & van Moerbeke, 1989b). That is, the solutions can be expressed in terms of Abelian integrals (a more thorough discussion of Abelian integrals and a definition of algebraic complete integrability will be given in Section 6.3). They proved that algebraic complete integrability constrains the solutions so that they can all be expanded in Laurent series. Moreover, they proved the converse: if the Laurent expansions form a "coherent tree" (that is, every expansion is related to the others in a tree-like structure), then the Hamiltonian is algebraically completely integrable.

The geometric approach was also adopted by Ercolani and Siggia. They considered Liouville integrable n-degree-of-freedom polynomial Hamiltonian systems whose Hamilton-Jacobi equation can be separated by changes of coordinates. They proved that such *hyperelliptically separable systems* enjoy the Painlevé property and that the order of the polynomial invariants are related to the positions in the local series where the arbitrary constants first appear.

3.8 Singularity analysis

The aim of singularity analysis is to build local solutions around the singularities of the solution in the complex plane. Although these local series expansions can be of different types depending on the system, the construction of the solution proceeds along the same lines. The information contained in these series will be addressed later on. The procedure described here is, at this stage, neither a test for integrability nor a way to explicitly solve the equation. Global statements on the equation such

[7] The form they considered is actually more general: $\dot{z} = J\frac{\partial H}{\partial z}$ where J is a skew symmetric matrix polynomial in z.

as the integrability properties or the existence of particular solutions will be inferred from a detailed analysis of the series expansions once they are built.

Singularity analysis proceeds in three different steps in order to express all possible series expansions around the movable singularities of the solutions. First, we obtain all possible dominant behaviors of the solution around a movable singularity. Second, for each possible dominant behavior, we find information on the arbitrary constants of the problems. That is, the position in the series expansion where arbitrary coefficients appear. Third, we build explicitly the series expansion up to the last arbitrary constants. Consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{3.49}$$

where $\mathbf{f}$ is an analytic vector field and assume that for a given choice of initial conditions there exists a singularity at the position $t_* \in \mathbb{C}$. Since $\mathbf{f} = \mathbf{f}(\mathbf{x})$ is autonomous[8], the singularity t_* is movable. That is, it depends on the choice of initial conditions. Singularity analysis consists of building local series around the singularity t_*. The most general type of formal series that can be built around the singularities is of the form

$$\mathbf{x}(t) = \tau^{\mathbf{p}}(\boldsymbol{\alpha} + \mathbf{P}(\tau^{\rho_1}, \ldots, \tau^{\rho_{n+1}})), \tag{3.50}$$

where $\tau = t - t_*$ and $\mathbf{P}$ is a vector of formal power series in the arguments with polynomial coefficients in $\log \tau$. The exponents $\rho_1, \ldots, \rho_{n+1}$ are complex numbers, that is,

$$\mathbf{x}(t) = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}(\log \tau) \tau^{\boldsymbol{\rho}.\mathbf{i}} \right), \qquad \boldsymbol{\rho}.\mathbf{i} = \sum_{j=1}^{n+1} \rho_j i_j. \tag{3.51}$$

This series can be obtained formally. However, it is not ordered in powers of τ in general. For instance if $n = 2$ and $\rho_1 = -\rho_2 = -1$, the contribution to the dominant term at the singularity of order τ^0 in the series is given by $\sum_{i=1}^{\infty} c_{i,i}$. In particular, if we want to check the absence of logarithmic terms in the series, we must compute infinitely many coefficients to each order in τ. Therefore, it is not convenient to use these series to obtain information about the behavior of the solutions in time. Thus, we will first restrict our analysis to the series associated with ascending powers of τ, that is, those associated with ρ with positive real parts. To do so, we consider series of the form

$$\begin{aligned} \mathbf{x}(t) &= \tau^{\mathbf{p}}(\boldsymbol{\alpha} + \mathbf{P}(\tau^{\rho_1}, \ldots, \tau^{\rho_k})), \\ &= \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}(\log \tau) \tau^{\boldsymbol{\rho}.\mathbf{i}} \right), \quad \boldsymbol{\rho}.\mathbf{i} = \sum_{j=1}^{k} \rho_j i_j. \end{aligned} \tag{3.52}$$

where $\Re(\rho_i) \geq 0 \ \forall\, i$ and $\mathbf{P}$ is a different vector of formal power series in the arguments with polynomial coefficients in $\log \tau$. We now consider in detail the formal construction of such series. The 3-step algorithm is based on the method of Ablowitz, Ramani and Segur (Ablowitz *et al.*, 1980a; Ablowitz *et al.*, 1980b).

3.8.1 Step 1: The dominant behavior

The first step consists of finding all the possible behaviors of the solution in the limit $t \to t_*$, where t_* is an arbitrary movable singularity. To do so, we find all possible weight-homogeneous decompositions

[8] A system is *autonomous* if it does not depend explicitly on time (*i.e.*, $\partial_t \mathbf{f}(\mathbf{x}) = \mathbf{0}$), *non-autonomous* otherwise.

of the vector field. Consider an analytic vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and assume, following Section 2.3.2, that there exists a decomposition

$$\mathbf{f} = \mathbf{f}^{(0)} + \mathbf{f}^{(1)} + \ldots + \mathbf{f}^{(m)}, \tag{3.53}$$

into $(m+1)$ weight-homogeneous components such that the *dominant part* of the vector field $\mathbf{f}^{(0)}$ is a scale-invariant system (as defined in Definition 2.13). That is, the system $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ supports the exact scale-invariant solution,

$$\mathbf{x}^{(0)} = \boldsymbol{\alpha}\tau^{\mathbf{p}}, \tag{3.54}$$

where $\tau = t - t_*$, $\mathbf{p} \in \mathbb{Q}^n$, $\boldsymbol{\alpha} \in \mathbb{C}^n$ and $|\boldsymbol{\alpha}| \neq 0$. With respect to the solution (3.54), the higher terms in the decomposition (3.53) are also weight-homogeneous:

$$f_i^{(j)}(t^{\mathbf{p}}\mathbf{x}) = t^{p_i+q^{(j)}-1} f_i^{(j)}(\mathbf{x}), \qquad i = 1, \ldots, n; \quad \forall\, t \in \mathbb{C}, \tag{3.55}$$

where $q^{(i)} \in \mathbb{Q}$ and

$$0 < q^{(i)} < q^{(j)} \quad \forall\, i < j. \tag{3.56}$$

Definition 3.3 A *dominant balance* (or *balance*) for a vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a pair $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\} \in \mathbb{C}^n \times \mathbb{Q}^n$ ($|\boldsymbol{\alpha}| \neq 0$) such that there exists a decomposition $\mathbf{f} = \sum_{i=0}^{m} \mathbf{f}^{(i)}$ with conditions (3.55-3.56).

The set of all possible dominant balances will be referred to as $\tilde{\mathcal{F}} = \{\{\boldsymbol{\alpha}^{(i)}, \mathbf{p}^{(i)}\},\ i = 1, \ldots, k\}$. The values of $\boldsymbol{\alpha}$ are obtained by finding the non vanishing solutions of

$$\mathbf{p}\boldsymbol{\alpha} = \mathbf{f}^{(0)}(\boldsymbol{\alpha}). \tag{3.57}$$

As a result, some of the components of $\boldsymbol{\alpha}$ may vanish. The *order of a balance* is the number of non-zero components of $\boldsymbol{\alpha}$. Different dominant balances, $\mathcal{F} \in \widetilde{\mathcal{F}}$, may define different decompositions of the vector fields. For each $\mathcal{F} \in \widetilde{\mathcal{F}}$, we proceed to Step 2.

Example 3.7 The three wave interaction. To illustrate this procedure, we consider a three-dimensional system (Kruskal *et al.*, 1990; Giacomini *et al.*, 1991) which is a variation of the model for the interaction of three quasi-synchronous waves in a plasma (2.7) given by

$$\dot{x} = -2y^2 + \gamma x + z - \delta^2, \tag{3.58.a}$$

$$\dot{y} = 2xy + \gamma y + \frac{\gamma\delta}{2}, \tag{3.58.b}$$

$$\dot{z} = -2zx - 2z, \tag{3.58.c}$$

where $x, y, z, \delta, \gamma \in \mathbb{C}$. For this system, we find three possible leading behaviors. First, consider the balances of order 2. They are obtained by considering the following decomposition of the vector field

$$\mathbf{f}^{(0)} = \begin{bmatrix} -2y^2 \\ 2xy \\ -2zx \end{bmatrix},\ \mathbf{f}^{(1)} = \begin{bmatrix} \gamma x \\ \gamma y \\ -2z \end{bmatrix},\ \mathbf{f}^{(2)} = \begin{bmatrix} -\delta^2/2 \\ \gamma\delta/2 \\ 0 \end{bmatrix},\ \mathbf{f}^{(3)} = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}. \tag{3.59}$$

The vector field $\dot{\mathbf{x}} = \mathbf{f}^{(0)}$ has two scale-invariant solutions which define two different balances

$$\mathbf{x}_{1,2} = \left(\frac{\tau^{-1}}{2}, \frac{\pm i\tau^{-1}}{2}, \alpha_3 \tau \right). \tag{3.60}$$

The balances $\mathcal{F}_{1,2} = \{(1/2, \pm i/2, \alpha_3), (-1, -1, 1)\}$ are both of order 2 since the third component of $\boldsymbol{\alpha}$ may vanish for some values of this arbitrary parameter. With respect to this solution, we define the non-dominant behaviors $q^{(i)}$ by the relation $\mathbf{f}^{(i)}(\tau^{\mathbf{p}}) \sim \tau^{\mathbf{p}+q^{(i)}-1}$ and find

$$q^{(1)} = 1, \quad q^{(2)} = 2, \quad q^{(3)} = 3. \tag{3.61}$$

The third balance is obtained by decomposing $\mathbf{f}$ as follows:

$$\mathbf{f}^{(0)} = \begin{bmatrix} z \\ 2xy + \gamma\delta/2 \\ -2zx \end{bmatrix}, \ \mathbf{f}^{(1)} = \begin{bmatrix} \gamma x \\ \gamma y \\ -2z \end{bmatrix}, \tag{3.62}$$

$$\mathbf{f}^{(2)} = \begin{bmatrix} -\delta^2/2 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{f}^{(3)} = \begin{bmatrix} -2y^2 \\ 0 \\ 0 \end{bmatrix}. \tag{3.63}$$

This decomposition is associated with the scale-invariant solution

$$\mathbf{x} = \left(\tau^{-1}, \frac{-\delta\gamma}{2}\tau, -\tau^{-2} \right). \tag{3.64}$$

The non-dominant exponents are

$$q^{(1)} = 1, \quad q^{(2)} = 2, \quad q^{(3)} = 4. \tag{3.65}$$

The balance $\mathcal{F}_3 = \{(1, -\frac{\delta\gamma}{2}, -1), (-1, 1, -2)\}$ is of order three (no vanishing coefficient) if $\delta\gamma \neq 0$. If $\delta\gamma = 0$, then $\mathcal{F}_3$ is of order 2 and reads $\mathcal{F}_3 = \{(1, 0, -1), (-1, 1, -2)\}$. Each of these balances will provide the first terms in an expansion around the singularity. ■

Example 3.8 Systems with no dominant behavior. Not all vector fields have a dominant behavior. Linear systems with constant coefficients do not have movable singularities since their solutions can be expressed in terms of products of exponentials and polynomials. Nonlinear systems can also fail to have dominant behavior of the type $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$. Consider, the system (Smith, 1973/74)

$$\dot{x} = y\rho^4 + \frac{a}{2}\rho^2 x + \frac{b}{4}x, \tag{3.66.a}$$

$$\dot{y} = -x\rho^4 + \frac{a}{2}\rho^2 y + \frac{b}{4}y, \tag{3.66.b}$$

where $\rho^2 = x^2 + y^2$. The vector field can be decomposed into homogeneous components, namely $\mathbf{f}^{(0)} = [y\rho^4, -x\rho^4]^{\mathrm{T}}$, $\mathbf{f}^{(1)} = [\frac{a}{2}\rho^2 x, \frac{a}{2}\rho^2 y]^{\mathrm{T}}$, and $\mathbf{f}^{(2)} = [\frac{b}{4}x, \frac{b}{4}y]^{\mathrm{T}}$. However, the system $\dot{\mathbf{x}} = \mathbf{f}^{(0)}$ does not admit a scale-invariant solution since the ansatz $x = \alpha\tau^{-1/4}, \ y = \beta\tau^{-1/4}$ leads to $\alpha = \beta = 0$. The absence of a scale-invariant solution can be understood on a particular case. For instance, if we take, $a = 0, \ b = 0$, the general solution is

$$x(t) = c\cos(c^2 t + \varphi), \tag{3.67.a}$$

$$y(t) = -c\sin(c^2 t + \varphi), \tag{3.67.b}$$

where c, φ are arbitrary constants. Hence, there is no movable singularity. However, even when the system exhibits singularities, it has no dominant behavior. For instance, if we take $a = 1,\ b = 0$, the solution

$$x(t) = (t_* - t)^{-1/2} \cos\left[c - (t_* - t)^{-1}\right], \tag{3.68.a}$$

$$y(t) = (t_* - t)^{-1/2} \sin\left[c - (t_* - t)^{-1}\right], \tag{3.68.b}$$

has an essential singularity. Finally, if we consider $a = 0$, $b = 1$, the solution

$$x(t) = ke^{t/4} \cos(c - k^4 e^t), \tag{3.69.a}$$

$$y(t) = ke^{t/4} \sin(c - k^4 e^t). \tag{3.69.b}$$

has no singularity in the complex time but spirals around the singularity at infinity. These examples show that in the absence of a dominant behavior, the type of singularity is not determined by the most nonlinear terms and the lower terms (in terms of weight) can drastically change the nature of the singularities. ■

Example 3.9 Another possibility of dominant behavior. Finally, even when there is an isolated movable singularity, this procedure might fail to detect it. Consider the third order system

$$\dddot{x} = \dot{x}(x^2 - \dot{x}^2). \tag{3.70}$$

The ansatz $x = \alpha\tau^p$ leads to two possible choices. First, $\ddot{x} = \dot{x}x^2$ may be dominant close to the singularity with $p = -1$. However, the term $\dot{x}^3 \sim t^{p+q-1}$ leads to $q = -4 < 0$ and we conclude that $\ddot{x} = \dot{x}x^2$ is not dominant. The second choice $\ddot{x} = \dot{x}^3$ leads to $p = 0$, which is a priori not admissible for a singularity. The dominant behavior at the movable singularities is actually $x = \pm\sqrt{2}\log(t - t_*)$. ■

3.8.2 Step 2: Kovalevskaya exponents

The Kovalevskaya exponents are a set of exponents associated with a given balance. They depend only on the dominant part $\mathbf{f}^{(0)}$ of the vector field. In the cases where the dominant balance defines the first term of a Laurent expansion, the Kovalevskaya exponents are the indices of the series coefficients where the arbitrary constants first appear. In general, they are defined as the Fuchs indices of the solutions of the variational equations. The variational equation of the dominant part of the vector field around the scale-invariant solution, $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$, for a balance $\mathcal{F}$ is given by

$$\dot{\mathbf{u}} = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}(t - t_*)^{\mathbf{p}})\mathbf{u}, \tag{3.71}$$

where $\mathbf{u} \in \mathbb{C}^n$ and $\left(D\mathbf{f}^{(0)}(\boldsymbol{\alpha})\right)_{ij} = \frac{\partial \mathbf{f}_i^{(0)}}{\partial x_j}(\boldsymbol{\alpha})$ is the Jacobian matrix evaluated on $\boldsymbol{\alpha}$. This linear system of equations has a set of fundamental solutions of the form:

$$\mathbf{u}^{(i)} = \boldsymbol{\gamma}^{(i)}(\log\tau)\tau^{\mathbf{p}+\rho_i}, \tag{3.72}$$

where $\boldsymbol{\gamma}^{(i)}$ is, in general, a polynomial function in $\log\tau$ and ρ_i is a *Kovalevskaya exponent*.[9] To compute the Kovalevskaya exponents, we introduce the *Kovalevskaya matrix* K:

$$K = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p}). \tag{3.73}$$

[9]The name *Kovalevskaya exponents* was first introduced by Yoshida (1983a; 1983b).

Definition 3.4 Let $\mathcal{F}$ be a dominant balance of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. The *Kovalevskaya exponents* $\mathcal{R} = \{\rho_1, \dots, \rho_n\}$ associated with $\mathcal{F}$ are the eigenvalues of the Kovalevskaya matrix $K = Df^{(0)}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p})$.

Proposition 3.3 *There exists a Kovalevskaya exponent equal to* -1 *with eigenvector* $\mathbf{f}^{(0)}(\boldsymbol{\alpha})$.

A balance is *principal* or *hyperbolic* if $(n-1)$ Kovalevskaya exponents have positive real parts.

Proof. The vector $\mathbf{u} = \mathbf{f}^{(0)}(\boldsymbol{\alpha}\tau^{\mathbf{p}})$ is a solution of the variational equation (3.71) as can be checked directly:

$$\begin{aligned} \dot{\mathbf{u}} &= D\mathbf{f}^{(0)}(\boldsymbol{\alpha}\tau^{\mathbf{p}})(\boldsymbol{\alpha}\mathbf{p}\tau^{\mathbf{p}-\mathbf{1}}), \\ &= \tau^{\mathbf{p}-\mathbf{2}} D\mathbf{f}^{(0)}(\boldsymbol{\alpha})(\boldsymbol{\alpha}\mathbf{p}). \end{aligned} \tag{3.74}$$

Since $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$ is a scale-invariant solution, we have $\mathbf{p}\boldsymbol{\alpha}\tau^{\mathbf{p}-\mathbf{1}} = \mathbf{f}^{(0)}(\boldsymbol{\alpha}\tau^{\mathbf{p}})$. Hence, $\dot{\mathbf{u}} = \tau^{\mathbf{p}-\mathbf{2}}\mathrm{diag}(\mathbf{p} - \mathbf{1})(\mathbf{p}\boldsymbol{\alpha})$. Substituting $\dot{\mathbf{u}}$ in (3.74), we obtain, after simplification

$$\left(D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p})\right)(\boldsymbol{\alpha}\mathbf{p}) = -\boldsymbol{\alpha}\mathbf{p}. \tag{3.75}$$

That is, $\boldsymbol{\beta} = \boldsymbol{\alpha}\mathbf{p}$ is an eigenvector of K with eigenvalue $\rho = -1$. □

Let $\rho_1 = -1$ and $\mathcal{R}^{(+)}$ be the set of of Kovalevskaya exponents with non-negative real parts. We also introduce the *Kovalevskaya eigenvectors* $\boldsymbol{\beta}^{(i)}$ and the *adjoint eigenvectors* $\bar{\boldsymbol{\beta}}^{(i)}$ where

$$K\boldsymbol{\beta}^{(i)} = \rho_i\boldsymbol{\beta}^{(i)}, \tag{3.76.a}$$

$$\bar{\boldsymbol{\beta}}^{(i)}K = \rho_i\bar{\boldsymbol{\beta}}^{(i)}. \tag{3.76.b}$$

We distinguish between the following cases.

1. K is semi-simple. Then, the eigenvectors corresponding to the same Kovalevskaya exponent are chosen to be orthogonal: that is, $\rho_i = \rho_j$, implies $\boldsymbol{\beta}^{(i)}.\boldsymbol{\beta}^{(j)} = 0$. In this case there exists a complete set of solutions for the variational equation of the form

$$\mathbf{u}^{(i)} = \boldsymbol{\beta}^{(i)}\tau^{\mathbf{p}+\rho_i}, \qquad i = 1, \dots, k < n. \tag{3.77}$$

2. If K is not semi-simple, there is no fundamental solution such as (3.77). Instead, logarithmic terms are introduced for the Kovalevskaya exponents whose geometric multiplicity is greater than the algebraic multiplicity. That is,

$$\mathbf{u}^{(i)} = \boldsymbol{\gamma}^{(i)}(\log\tau)\tau^{\mathbf{p}+\rho_i}, \qquad i = 1, \dots, n, \tag{3.78}$$

where $\boldsymbol{\gamma}^{(i)}(\log(t - t_*))$ is a polynomial in $\log(t - t_*)$ of degree equal to the difference between the algebraic and the geometric multiplicity of the Kovalevskaya exponent, ρ_i.

There exists another set of exponents closely related to the Kovalevskaya exponents, the so-called *resonances* (Ablowitz *et al.*, 1980a). The resonances are defined in a similar fashion but the solution $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$ around which they are defined is chosen such that $\alpha_i \neq 0 \ \forall \ i$. It has been shown that resonances and Kovalevskaya exponents are, in general, different (Yoshida *et al.*, 1987b). More precisely, they are different whenever the order of the dominant balance is less than the number n of independent variables.

Example 3.10 Continuation of Example (3.58). For each balance $\mathcal{F}_i$, we compute the variational equation and the associated Kovalevskaya matrix K_i. For $\mathcal{F}_{1,2}$, we obtain

$$K_{1,2} = \begin{bmatrix} 1 & \mp 2i & 0 \\ \pm i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.79}$$

with associated eigenvalues $\mathcal{R} = \{-1, 0, 2\}$ and eigenvectors

$$\boldsymbol{\beta}^{(1)} = (1, \mp i, 0),\ \boldsymbol{\beta}^{(2)} = (0, 0, 1),\ \boldsymbol{\beta}^{(3)} = (\mp 2i, 1, 0). \tag{3.80}$$

For $\mathcal{F}_3$ and $\delta\gamma \neq 0$ we obtain

$$R_3 = \begin{bmatrix} 1 & 0 & 1 \\ -\delta\gamma & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \qquad \mathcal{R} = \{-1, 1, 2\}, \tag{3.81}$$

$$\boldsymbol{\beta}^{(1)} = (1, \delta\gamma, -2),\ \boldsymbol{\beta}^{(2)} = (0, 1, 0),\ \boldsymbol{\beta}^{(3)} = (1, -\delta\gamma, 1). \tag{3.82}$$

For $\delta\gamma = 0$ we have $\mathcal{R} = \{-1, 0, 2\}$. ■

Example 3.11 Irrational Kovalevskaya exponents. Kovalevskaya exponents can be functions of the parameters of the vector field. For instance, the planar Lotka-Volterra system

$$\dot{x} = x(ay - x), \tag{3.83.a}$$
$$\dot{y} = y(bx - y), \tag{3.83.b}$$

is homogeneous, hence $\mathbf{p} = (-1, -1)$ and $\boldsymbol{\alpha}$ is given by the solution of

$$\alpha_1 = \alpha_1(a\alpha_2 - \alpha_1), \tag{3.84.a}$$
$$\alpha_2 = \alpha_2(b\alpha_1 - \alpha_2), \tag{3.84.b}$$

which leads to the three different balances

$$\mathcal{F}_1 = \{\boldsymbol{\alpha} = (1, 0), \mathbf{p} = (-1, -1)\}, \tag{3.85.a}$$
$$\mathcal{F}_2 = \{\boldsymbol{\alpha} = (0, 1), \mathbf{p} = (-1, -1)\}, \tag{3.85.b}$$
$$\mathcal{F}_3 = \{\boldsymbol{\alpha} = (\frac{1+b}{1-ab}, \frac{1+a}{1-ab}), \mathbf{p} = (-1, -1)\}. \tag{3.85.c}$$

The corresponding sets of Kovalevskaya exponents are, respectively,

$$\mathcal{R}_1 = \{-1, 1 + b\}, \tag{3.86.a}$$
$$\mathcal{R}_2 = \{-1, 1 + a\}, \tag{3.86.b}$$
$$\mathcal{R}_3 = \{-1, \frac{(b+1)(a+1)}{(ab-1)}\}. \tag{3.86.c}$$

The second exponent ρ depends on the coefficients of the vector field and can assume, a priori, any real value, for real a, b. ■

For a planar vector field since $\rho = -1$ is always a Kovalevskaya exponent, the second Kovalevskaya exponent is rational in the coefficients of the vector field. However, for n-dimensional vector fields with $n > 2$, the Kovalevskaya exponents are the roots of a polynomial of degree $(n - 1)$ and can become complex. The relationship between Kovalevskaya exponents and the degree of first integrals will be explored in Chapter 5.

3.8.3 Step 3: The local solution

For each different dominant balance, we can compute the Kovalevskaya exponents. It is now possible to build a formal expansion of the solutions around the singularities. Different cases should be considered depending on the rationality of the Kovalevskaya exponents.

Formal series

For a given system, a general formal expansion of the the general solution can be built around its singularities. Consider again a balance $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$ and let $\mathcal{R} = \{\rho_1, \dots, \rho_n\}$ be the set of Kovalevskaya exponents. Then, there exists a formal solution:

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{\mathbf{i},\ |\mathbf{i}|>1}^{\infty} \mathbf{c}_{\mathbf{i}} \tau^{\boldsymbol{\rho}.\mathbf{i}} \right), \qquad \boldsymbol{\rho}.\mathbf{i} = \sum_{j=1}^{n+1} \rho_j i_j, \tag{3.87}$$

where $\mathbf{c}_{\mathbf{i}}$ is polynomial in $\log \tau$ and $1/\rho_{n+1}$ is the least common denominator of $\{q^{(1)}, \dots, q^{(m)}\}$.

The coefficients $\mathbf{c}_{\mathbf{i}}$ can be computed by a linear recursion relation built on the matrix K by considering increasing values of $|\mathbf{i}|$. The expansion involves as many arbitrary constants as the number of exponents (taking into account the algebraic multiplicity of each Kovalevskaya exponent) and its formal existence will be proved in Section 3.8.4. These formal series are not ordered in increasing powers of τ since $\boldsymbol{\rho}.\mathbf{i}$ can become negative. Therefore, different cases have to be considered depending on the different types of branching the solution may exhibit. It will be shown that a general condition required by most integrability theories is the absence of logarithmic branch points.

We now discuss, in order of complexity, different types of local series with ascending powers.

The Puiseux series

Consider the balance $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$. We start with the case where all the Kovalevskaya exponents with positive real parts are rational. Let $\mathcal{R}^{(+)}$ be the set of all such Kovalevskaya exponents. The simplest expansion that we can build is a *Puiseux series*, that is, a series containing only rational powers of $(t - t_*)$. To do so, we define $1/s$ to be the least common denominator of the set $S = \{q^{(1)}, \dots, q^{(m)}\} \cup \mathcal{R}^{(+)}$ and we look for a solution in terms of a *Puiseux series* containing as many free parameters as the number of positive Kovalevskaya exponents. That is,

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{i=0}^{\infty} \mathbf{c}_i \tau^{i/s} \right), \tag{3.88}$$

where $\mathbf{c}_i \in \mathbb{C}^n \ \forall\, i$. From the analysis of Kovalevskaya exponents, we know that new arbitrary constants will appear in the series for the coefficients $\mathbf{c}_h$ where $h = \rho s,\ \rho \in \mathcal{R}^{(+)}$. These arbitrary constants may introduce incompatible constraints on the coefficients $\mathbf{c}_j$ $(j < h)$. The coefficients are computed by inserting the full Puiseux series (3.88) into the original system (3.49) and by explicitly determining the recursion relation for the coefficients $\mathbf{c}_j$:

$$K\mathbf{c}_j = \frac{j}{s}\mathbf{c}_j - \mathbf{P}_j(\mathbf{c}_1, \dots, \mathbf{c}_{j-1}), \tag{3.89}$$

where $\mathbf{P}_j$ is polynomial in its variables.

If $j/s = \rho$ is an eigenvalue of K (that is, a Kovalevskaya exponent), we have to impose two conditions on the parameter of the system to solve the linear system for $\mathbf{c}_h$ and obtain as many free parameters as the algebraic multiplicity of the Kovalevskaya exponents.

1. The Kovalevskaya matrix K must be semi-simple, else it would not be possible to introduce as many arbitrary constants as necessary to describe the solution and logarithmic terms should be introduced.

2. We use the Fredholm alternative to find conditions for the existence of a solution. That is, we apply $\bar{\boldsymbol{\beta}}$ on (3.89) to obtain

$$C_\rho \equiv \bar{\boldsymbol{\beta}}.\mathbf{P}_j = 0 \quad \forall\, \rho \in \mathcal{R}^{(+)}. \tag{3.90}$$

   This relation must hold for all eigenvectors $\bar{\boldsymbol{\beta}}$ of K^T of eigenvalue ρ.

The above two conditions are central in the singularity analysis and are usually referred to as *compatibility conditions* . If both conditions are fulfilled, then there exists a local Puiseux solution with as many arbitrary parameters as the number of positive Kovalevskaya exponents.

Logarithmic expansions

If one of the above conditions is not satisfied, then it is not possible to find a series solution (3.88) which only includes powers of $(t - t_*)^{1/s}$. Therefore, the initial ansatz is not correct and in order to restore arbitrariness, logarithmic terms should be introduced in the series. For simplicity, assume that for the first positive exponent ρ, the compatibility condition is not satisfied, that is, $C_\rho \neq 0$, and $C_{\rho_i} = 0\ \forall\ \rho_i \neq \rho$. Then, it is possible to find a consistent expansion involving logarithmic terms of the form

$$\mathbf{x} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbf{c}_{ij}\tau^{\mathbf{p}+i/s}(\tau^\rho \log \tau)^{j/s}. \tag{3.91}$$

Any series containing log-terms will be called a Ψ*-series* (Bender & Orszag, 1978). It is now possible to set the value of coefficient $\mathbf{c}_{01}$ in order to restore the arbitrariness of coefficient $\mathbf{c}_{h0}$. All other coefficients now can be determined recursively.

Example 3.12 The Duffing Oscillator. The two-dimensional system

$$\dot{x} = y, \tag{3.92.a}$$
$$\dot{y} = 3ay + 2bx - 2x^3. \tag{3.92.b}$$

has a unique weight-homogeneous decomposition given by

$$\mathbf{f}^{(0)} = \begin{bmatrix} y \\ -2x^3 \end{bmatrix},\ \mathbf{f}^{(1)} = \begin{bmatrix} 0 \\ 3ay \end{bmatrix},\ \mathbf{f}^{(2)} = \begin{bmatrix} 0 \\ 2bx \end{bmatrix}. \tag{3.93}$$

There are two possible balances given by the scale-invariant solutions of $\ddot{x} = -2x^3$, with $\boldsymbol{\alpha} = (\pm i, \mp i)$, $\mathbf{p} = (-1, -2)$, $q^{(1)} = 1$, and $q^{(2)} = 2$. For both balances, the Kovalevskaya matrix reads

$$K = \begin{bmatrix} 1 & 1 \\ 6 & 2 \end{bmatrix}, \tag{3.94}$$

with eigenvalues $\mathcal{R} = \{-1, \rho = 4\}$. We can now build the local series

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} i\tau^{-1} \\ -i\tau^{-2} \end{bmatrix} + \begin{bmatrix} c_1\tau^0 \\ d_1\tau^{-1} \end{bmatrix} + \begin{bmatrix} c_2\tau^1 \\ d_2\tau^0 \end{bmatrix} + \begin{bmatrix} c_3\tau^2 \\ d_3\tau^1 \end{bmatrix} + \begin{bmatrix} c_4\tau^3 \\ d_4\tau^2 \end{bmatrix} + \dots \tag{3.95}$$

The largest exponent is $\rho = 4$. Therefore, we compute the coefficients c_4, d_4 explicitly to prove the existence of the Laurent series. If the compatibility condition C_4, given by (3.90), is not satisfied, we modify the ansatz (3.95) to include logarithmic terms. The coefficients $\mathbf{c}_j = (c_j, d_j)$ are obtained by the recursion relation

$$K\mathbf{c}_j = j\mathbf{c}_j + \mathbf{P}_j(\mathbf{c}_1, \dots, \mathbf{c}_{j-1}). \tag{3.96}$$

For all $j < \rho$, there is a unique solution $\mathbf{c}_j = (K - j\mathrm{I})^{-1}\mathbf{P}_j$. The explicit form of the first few coefficients are

$$\mathbf{c}_1 = \begin{bmatrix} \frac{ia}{2} \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -\frac{i}{12}(4b + 3a^2) \\ -\frac{i}{12}(4b + 3a^2) \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} ia(b + a^2) \\ \frac{ia}{2}(b + a^2) \end{bmatrix}. \tag{3.97}$$

However, for $j = \rho = 4$, the solution of (3.96) may not exist and we have a compatibility condition obtained by applying the eigenvector $\bar{\boldsymbol{\beta}}_4 = (2, 1)$ of K^T of eigenvalue 4 to $\mathbf{P}_4 = (0, -6ia^2(b + a^2))$. That is,

$$C_4 = \bar{\boldsymbol{\beta}}_4.\mathbf{P}_4 = -6ia^2(b + a^2). \tag{3.98}$$

Therefore, in order for the Duffing system to have meromorphic solutions, the condition $C_4 = 0$ should be satisfied (the same condition is obtained for the other balance). That is, either $a = 0$ in which case, the system is Hamiltonian with $H = \frac{1}{2}y^2 - bx^2 + \frac{1}{2}x^4$; or $b + a^2 = 0$, in which case, there exists a time-dependent first integral $I = (y^2 + x^4 + a^2x^2 - 2axy)\exp(-4at)$.

If $a(b + a^2) \neq 0$, the system does not admit a Laurent expansion and the ansatz (3.95) has to be modified to include logarithmic terms. In this case, the series reads

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbf{c}_{ij}\tau^i(\tau^4 \log \tau)^j \right). \tag{3.99}$$

The coefficients $\mathbf{c}_{00} = 0$, $\mathbf{c}_{i0} = \mathbf{c}_i$, $i = 1, 2, 3$ are given by (3.97). But to order $\tau^{\mathbf{p}+4}$, we have the two systems

$$K\mathbf{c}_{01} = 4\mathbf{c}_{01}, \tag{3.100}$$

and

$$K\mathbf{c}_{40} = 4\mathbf{c}_{40} - \mathbf{c}_{01} + \mathbf{P}_4. \tag{3.101}$$

The compatibility condition now reads

$$C_4 = \bar{\boldsymbol{\beta}}_4.(-\mathbf{c}_{01} + \mathbf{P}_4) = 0. \tag{3.102}$$

Equation (3.100) leads to $\mathbf{c}_{01} = \kappa_1\boldsymbol{\beta}_4 = \kappa_1(1, 3)$, where κ_1 is an arbitrary constant set by the Fredholm condition (3.102): $-\kappa_1\bar{\boldsymbol{\beta}}_4.\boldsymbol{\beta}_4 - 6ia^2(b + a^2) = 0$, that is, we choose $\kappa_1 = \frac{6}{5}ia^2(b + a^2)$ and the system for $\mathbf{c}_{4,0}$ has the solution

$$\mathbf{c}_{40} = \kappa\boldsymbol{\beta}_4 + \begin{bmatrix} 0 \\ \frac{6}{5}ia^2(b + a^2) \end{bmatrix}. \tag{3.103}$$

The rest of the series can now be computed recursively on the two indices i, j. ■

Example 3.13 The three wave system revisited. We compute the compatibility conditions for each balance of the system (3.58). That is, we insert the expansion $\mathbf{x} = \boldsymbol{\alpha} \sum_{i=0}^{\infty} \mathbf{c}_i (t - t_*)^{i+\mathbf{p}}$ in system (3.58) and compute for a given $\{\boldsymbol{\alpha}, \mathbf{p}\}$ the coefficients $\mathbf{c}_i$ by the recursion relation (3.89). To prove the existence of a formal Laurent expansion, the coefficients $\mathbf{c}_i$ are computed only up to $i = \rho_{\max}$.

For the balance $\mathcal{F}_3$ and $\delta\gamma \neq 0$, we obtain two compatibility conditions, one for the Kovalevskaya exponent $\rho = 1$, and the other for the Kovalevskaya exponent $\rho = 2$. They are

$$C_1 = \bar{\boldsymbol{\beta}}_2.P_2 = 2, \tag{3.104.a}$$
$$C_2 = \bar{\boldsymbol{\beta}}_2.P_3 = -\delta^2. \tag{3.104.b}$$

The balance $\mathcal{F}_3$ does not define the leading terms of a Laurent series since $C_1 \neq 0$. However, if we set $\delta\gamma = 0$, we obtain

$$C_0 = 0, \tag{3.105.a}$$
$$C_2 = \begin{cases} -2\gamma(\gamma+1) & \text{if } \delta = 0, \\ -\delta^2 & \text{if } \gamma = 0. \end{cases} \tag{3.105.b}$$

Unless $\delta = 0$ and $\gamma(\gamma+1) = 0$, no Laurent expansion exists for the solution starting with the dominant balance $\mathcal{F}_3$. We still have to check the two balances $\mathcal{F}_{1,2}$. For these, the compatibility conditions associated with $\rho = 0$ vanish identically and $C_2 = \frac{\delta^2 + i\delta\gamma}{2}$.

We conclude that a necessary condition for the system to be free of logarithmic branch points is $\delta = 0$ and $\gamma(\gamma+1) = 0$. In the first case, $\delta = \gamma = 0$, the system is integrable with one first integral

$$zy = \kappa_1 e^{-2t}. \tag{3.106}$$

For the second case, $\delta = \gamma + 1 = 0$, the system has the first integral

$$x^2 + (y + \frac{\delta}{2})^2 + z = \kappa_2 e^{-2t}. \tag{3.107}$$

In either case, the general solution can be written in terms of elliptic integrals. ■

3.8.4 Formal existence of local solutions

In the previous section, we built local series with ascending powers in the particular cases where the Kovalevskaya exponents are rational. We now consider the general problem of proving the formal existence of local solutions around the movable singularities for the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{C}^n, \tag{3.108}$$

where $\mathbf{f}$ is analytic. Assume that there exists a dominant balance $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$. That is, there exists a weight-homogeneous decomposition of the vector field

$$\mathbf{f} = \sum_{i=0}^{m} \mathbf{f}^{(i)}, \tag{3.109}$$

such that the system $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ supports an exact scale-invariant solution $\mathbf{x}^{(0)} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$, where $\tau = t - t_*$, $\mathbf{p} \in \mathbb{Q}^n$, $\boldsymbol{\alpha} \in \mathbb{C}^n$ and $|\boldsymbol{\alpha}| \neq 0$. Moreover, the higher order terms in the decomposition (3.109) are also weight-homogeneous:

$$f_i^{(j)}(t^{\mathbf{p}}\mathbf{x}) = t^{p_i + q^{(j)} - 1} f_i^{(j)}(\mathbf{x}), \qquad i = 1, \dots, n, \quad \forall\, t \in \mathbb{C}_0, \tag{3.110}$$

where $q^{(i)} \in \mathbb{Q}$ and $q^{(i)} > 0\ \forall\, i$. Let $1/q$ be the least common denominator of the set $\{q^{(1)}, \dots, q^{(m)}\}$ and let the Kovalevskaya exponents $\{\rho_1, \dots, \rho_n\}$ be the eigenvalues of the matrix $K = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \text{diag}(\mathbf{p})$. The following theorem guarantees the existence of formal solutions.

Theorem 3.2 *Consider the analytic vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, with $\mathbf{x} \in \mathbb{C}^n$. Assume that there is a dominant balance $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$. Then, there exists a formal local solution with n arbitrary constants of the form*

$$\mathbf{x}(t) = t^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}(\log t) t^{\boldsymbol{\rho}.\mathbf{i}} \right), \qquad \boldsymbol{\rho}.\mathbf{i} = \sum_{j=k}^{n+1} \rho_j i_j, \tag{3.111}$$

where $\mathbf{c}_{\mathbf{i}}(\log t)$ is polynomial in $\log t$, $\{\rho_1, \dots, \rho_n\}$ are the eigenvalues of the Kovalevskaya matrix K and $\rho_{n+1} = q$.

To prove this theorem we will need the following Lemma.

Lemma 3.1 *Let R be an $n \times n$ matrix with constant complex coefficients and $\mathbf{u}(\log t)$ a vector of polynomials in $\log t$. Then, the general solution of the linear system*

$$t\dot{\mathbf{y}} = R\mathbf{y} + \mathbf{u}(\log t), \tag{3.112}$$

is

$$\mathbf{y} = t^R \mathbf{C} + \mathbf{v}(\log t), \tag{3.113}$$

where t^R is the exponential matrix given by $t^R = e^{(\log t)R}$, $\mathbf{C}$ is a vector of arbitrary constants and $\mathbf{v}(\log t)$ is a vector of polynomials in $\log t$ given by

$$\mathbf{v}(\log t) = t^R \int^t s^{-R-1} \mathbf{u}(\log s) ds. \tag{3.114}$$

Moreover, if R is semi-simple then $t^R\mathbf{C} = \sum_{i=1}^k \boldsymbol{\beta}^{(i)} t^{r_i}$ where $\boldsymbol{\beta}^{(i)}$ is a vector spanning the eigenspace of R of eigenvalue r_i (with as many arbitrary constants as the dimension of the eigenspace). If R is not semi-simple then $t^R = \sum_{i=1}^k \boldsymbol{\beta}^{(i)} t^{r_i}$, and $\boldsymbol{\beta}^{(i)}$ is polynomial in $\log t$ and can be expressed in terms of the generalized eigenvectors of R.[10]

Proof. (Lemma) First consider the homogeneous system $t\dot{\mathbf{y}} = R\mathbf{y}$. By direct computation, we verify that $\mathbf{y} = t^R\mathbf{C}$ is the general solution ($\mathbf{C}$ is a vector of arbitrary constants). Equivalently, this solution can be written $t^R\mathbf{C} = \sum_{i=1}^k \boldsymbol{\beta}^{(i)} t^{r_i}$. If R is semi-simple, $\boldsymbol{\beta}^{(i)}$ is a vector spanning the eigenspace of R of eigenvalue r_i. If R is not semi-simple, then $\boldsymbol{\beta}^{(i)}$ is a polynomial in $\log t$ constructed from the

[10] If ρ is an eigenvalue of R of multiplicity $m \leq n$. Then, for $k = 1, \dots, m$, any nonzero solution $\mathbf{v}$ of $(R - \rho \mathrm{I})^k \mathbf{v} = \mathbf{0}$ is a *generalized eigenvector* of R.

generalized eigenvectors of R of eigenvalue r_i. The method of variation of constants on the solution $\mathbf{y} = t^R\mathbf{C}(t)$ leads to

$$t\dot{\mathbf{C}} = t^R\mathbf{u}(\log t), \tag{3.115}$$

whose solution is

$$\mathbf{C} = \int^t s^{-R-1}\mathbf{u}(\log s)ds + \tilde{\mathbf{C}}. \tag{3.116}$$

Hence, the general solution is

$$\mathbf{y} = t^R\mathbf{C} + t^R\int^t s^{-R-1}\mathbf{u}(\log s)ds. \tag{3.117}$$

To prove that $\mathbf{v} = t^R\int^t s^{-R-1}\mathbf{u}(\log s)ds$ is polynomial in $\log\tau$, we compute the integral by parts and use induction to show that $\mathbf{v}$ is polynomial in $\log\tau$ of degrees equal to the sum of the degrees of $\mathbf{u}(\log s)$ and the number of vanishing eigenvalues of R. □

Proof. (Theorem) We substitute the general form of the solution for $\mathbf{x}$ in the system, and show that for increasing values of $\mathbf{i}$, we can solve iteratively for $\mathbf{c_i}$. First, we compute $\dot{\mathbf{x}}$. That is,

$$\dot{\mathbf{x}} = t^{\mathbf{p-1}}\left[\mathbf{p}\boldsymbol{\alpha} + \sum_{\mathbf{i}} t^{\boldsymbol{\rho}.\mathbf{i}}\left(t\dot{\mathbf{c}}_\mathbf{i} + \mathbf{c_i}(\mathbf{p} + \boldsymbol{\rho}.\mathbf{i})\right)\right]. \tag{3.118}$$

We substitute the solution in the vector field and use the fact that there exists $m \in \mathbb{N}$ such that $\mathbf{f}(t^\mathbf{p}\mathbf{y}) = \sum_{i=0}^m t^{\mathbf{p}-1+iq}\mathbf{f}^{(i)}(\mathbf{y})$. The Taylor expansion of $\mathbf{f}$ around the scale-invariant solution gives

$$\mathbf{f}(\mathbf{x}) = t^{\mathbf{p}-1}\left[\mathbf{f}^{(0)}(\boldsymbol{\alpha}) + \sum_{\mathbf{i}} D\mathbf{f}^{(0)}(\boldsymbol{\alpha})\mathbf{c_i}t^{\boldsymbol{\rho}.\mathbf{i}} + \sum_{\mathbf{i}} \mathbf{d_i}t^{\boldsymbol{\rho}.\mathbf{i}}\right], \tag{3.119}$$

where $\mathbf{d_i} = \mathbf{d_i}(\mathbf{c_j}, |\mathbf{j}| < |\mathbf{i}|)$. We can compare (3.118) with (3.119) to obtain, after simplification,

$$t\dot{\mathbf{c}}_\mathbf{i} = K_\mathbf{i}\mathbf{c_i} + \mathbf{d_i}, \tag{3.120}$$

where $K_\mathbf{i} = K - (\boldsymbol{\rho}.\mathbf{i})\mathrm{I}$. We want to prove that $\mathbf{c}_i$ is a function of $\log t$ only. Hence, we look for solutions of (3.120) that are either constant or polynomial in $\log t$. Depending on the vector $\mathbf{i}$, we consider the following two cases. First, if $\boldsymbol{\rho}.\mathbf{i} \neq \rho_j$ for $j = 1, \ldots, n$, then the matrix $K_\mathbf{i}$ is invertible and $\mathbf{c_i} = K_\mathbf{i}^{-1}\mathbf{d_i}$. Second, if

$$\boldsymbol{\rho}.\mathbf{i} = \rho_j, \tag{3.121}$$

for some $1 \leq j \leq n$, then there exists a solution provided by Lemma 3.1 of the form

$$\mathbf{c_i} = \boldsymbol{\beta}^{(j)} + \mathbf{v}(\log t), \tag{3.122}$$

where $\mathbf{v}(\log t)$ is given by (3.114) with $\mathbf{u} = \mathbf{d_i}$. If K is semi-simple then $\boldsymbol{\beta}^{(j)}$ is an arbitrary sum of eigenvectors spanning the eigenspace of K of eigenvalue ρ_j. If K is not semi-simple, then $\boldsymbol{\beta}^{(j)}$ is polynomial in $\log t$ built from the generalized eigenvectors of K of eigenvalue ρ_j. In both cases, the number of arbitrary parameters in $\boldsymbol{\beta}^{(j)}$ equals the algebraic multiplicity of ρ_j, and $\mathbf{c_i}$ is polynomial in $\log t$. We conclude that all the coefficients $\mathbf{c_i}$ are either constant or polynomial in $\log t$. Moreover, since the sum starts at $|\mathbf{i}| = 1$, there are at least n equalities $\boldsymbol{\rho}.\mathbf{i} = \rho_j$, which implies that there are n free parameters in the solution. □

3.8.5 Companion systems

The construction of the local series around the singularities is, in many ways, reminiscent of the construction of local exponential series around a fixed point. To make this analogy transparent, we introduce an explicit transformation that maps the local series around the singularities to the local series around a fixed point of a new system, the *companion system*.

Local analysis around fixed points

We briefly recall the basic ingredients of the local analysis of vector fields. Consider the system of differential equations

$$\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{C}^n, \tag{3.123}$$

where $\mathbf{f}$ is analytic. Assume that this system has k isolated fixed points $\{\mathbf{x}^{(i)},\ i = 1, \ldots, k\}$. For each fixed point, $\mathbf{x}^{(i)} = \mathbf{x}_*$, we introduce $\mathbf{y} = \mathbf{x} - \mathbf{x}_*$ and consider the system

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) = \mathbf{f}(\mathbf{y} + \mathbf{x}_*), \quad \mathbf{y} \in \mathbb{C}^n. \tag{3.124}$$

Therefore, we obtain k systems that can be analyzed locally around the origin. For each system, the vector $\mathbf{g}$ can be split into two parts:

$$\mathbf{g} = \mathbf{g}^{\text{lin}} + \mathbf{g}^{\text{nli}} \tag{3.125}$$

where $\mathbf{g}^{\text{lin}}$ and $\mathbf{g}^{\text{nli}}$ are, respectively, the *linear* and *nonlinear* parts of the vector field. The spectrum of the Jacobian matrix $D\mathbf{g}(\mathbf{0}) = D\mathbf{g}^{\text{lin}}$ defines the *linear eigenvalues* $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$ (where the real parts of the λ_i's are assumed to be in increasing order). The fixed point is said to be *hyperbolic* if $\Re(\lambda_i) \neq 0,\ i = 1, \ldots, n$.

Locally, around the origin, there exists a formal local series solution of the form

$$\mathbf{y}(t) = \mathbf{P}(C_1 e^{\lambda_1 t}, \ldots, C_n e^{\lambda_n t}), \tag{3.126}$$

where $\mathbf{P}$ is a vector of formal power series in the arguments with polynomial coefficients in t and where $C_1, \ldots, C_n$ are arbitrary constants. For instance, along the unstable manifold $W_u(\mathbf{0})$, we have, taking $C_i = 0$ for all i such that $\Re(\lambda_i) \leq 0$,

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{P}_u(C_k e^{\lambda_k t}, \ldots, C_n e^{\lambda_n t}), \\ &= \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}(t)(\mathbf{C}^{\text{u}})^{\mathbf{i}} e^{(\boldsymbol{\lambda}^{\text{u}}.\mathbf{i})t}, \qquad \boldsymbol{\lambda}^{\text{u}}.\mathbf{i} = \sum_{j=k}^{n} \lambda_j i_j \end{aligned} \tag{3.127}$$

This series is a local expansion of the $(n-k+1)$ parameter solutions on the local unstable manifold. The coefficients $\mathbf{c}_{\mathbf{i}}(t)$ are polynomials in t of degree less than or equal to $\boldsymbol{\lambda}^{\text{u}}.\mathbf{i}$. If all $\mathbf{c}_i$'s are constant, then the series is referred to as a *pure series*, that is, a pure series in exponentials. If all the coefficients $\mathbf{c}_{\mathbf{i}}$ are constant for $|\mathbf{i}| \leq \lambda_n$, then they are constants for all $\mathbf{i}$. The convergence of this series is guaranteed by the unstable manifold theorem (Chow & Hale, 1982, p. 103).

Lemma 3.2 *Let* $\mathbf{y} = \mathbf{0}$ *be a fixed point of* $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$ *with linear eigenvalues* $\boldsymbol{\lambda}$*. Let* $\boldsymbol{\lambda}^u$ *be the vector of linear eigenvalues with positive real parts and* $\mathbf{y}(t; \mathbf{C}^u)$ *the series (3.127). Then, there exist*

$K, t_1, \alpha \in \mathbb{R}$, *with* $0 < \alpha < \Re(\lambda_i^u)$, *and* $i = k, \ldots, N$, *and* δ *small enough such that for* $|\mathbf{a}| < \delta$, $t < t_1$ *we have*

$$|\mathbf{y}(t;\mathbf{a})| < K|\mathbf{a}|e^{\alpha t}. \tag{3.128}$$

The prefactor K can be explicitly related to the norm of $\exp(A^{\mathrm{u}}t)$ where A^{u} is the Jordan block associated with the linear eigenvalues $\boldsymbol{\lambda}^{\mathrm{u}}$.

Companion transformation

There is an obvious parallel between the analysis of the solutions around the fixed points in phase space and the analysis around the movable singularities (underlined by the analogy between the series solutions (3.126-3.127) and (3.111)). In more than one way, the construction of the series (3.111) is similar to the construction of the solutions (3.127) on the unstable manifold. To further develop this analogy, we introduce a change of variables which transforms the local analysis around the singularities into the local analysis around the fixed points of another system. Consider the balance $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$ together with the decomposition of the vector field $\mathbf{f}$:

$$\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{f}^{(i)}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{C}^n, \tag{3.129}$$

and define the transformation $T : (\mathbf{x}, t) \to (\mathbf{X}, s)$ by

$$\mathbf{x}(t) = \tau^{\mathbf{p}}\mathbf{X}(\tau), \tag{3.130.a}$$

$$\tau = e^s. \tag{3.130.b}$$

The *companion system* is

$$X_i' = F_i(X_1, \ldots, X_{n+1}), \qquad i = 1, \ldots, n, \tag{3.131.a}$$

$$X_{n+1}' = qX_{n+1}, \qquad \text{that is, } X_{n+1} = e^{qs}, \tag{3.131.b}$$

where $1/q \in \mathbb{N}$ is be the least common denominator of the set $\{q^{(1)}, \ldots, q^{(m)}\}$ as defined in (3.111). We will use upper case to denote the dependent variables and vector fields of the companion system. In compact form, this system reads

$$\mathbf{X}' = \mathbf{F}(X_1, \ldots, X_N), \quad \mathbf{X} \in \mathbb{C}^N, \ N = n + 1. \tag{3.132}$$

In general, the embedding $n \to n + 1$ is necessary to rewrite the new system as an autonomous vector field. However, in the particular case where the system is *weight-homogeneous* (that is, $\mathbf{f} = \mathbf{f}^{\mathrm{up}}$), the embedding is not necessary since F_i $(i = 1, \ldots, n)$ does not depend on X_{n+1} and the last equation (3.131.b) decouples. Since a companion system can be defined for each balance, there are in general k companion systems associated with a given system. Some of these systems are actually identical (for balances with equal vectors $\mathbf{p}$ but different $\boldsymbol{\alpha}$). The companion system is, for real $\mathbf{p}$, a real analytic vector field. However, we will carry out its analysis around complex fixed points (whenever $\boldsymbol{\alpha} \in \mathbb{C}^n$). The transformation to companion systems has been used independently by different authors (Delshams & Mir, 1997; Furta, 1996) in the context of the Painlevé analysis (see also Costin and Costin (1997; 1998) and Goriely (2001)).

Example 3.14 A simple example of companion systems. Consider the planar vector field

$$\dot{x}_1 = x_2, \tag{3.133.a}$$
$$\dot{x}_2 = 4x_1^3 - 2a. \tag{3.133.b}$$

The possible dominant balances are $\mathbf{p} = (-1, -2)$ with $\boldsymbol{\alpha} = (\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2})$ and $q = 1$. That is,

$$x_1 = \alpha_1 \tau^{-1}, \quad x_2 = \alpha_2 \tau^{-2}, \tag{3.134}$$

is an exact solution of $\dot{\mathbf{x}} = \mathbf{f}^{(0)}$ with $\mathbf{f}^{(0)} = [x_2, 4x_1^3]^{\mathrm{T}}$ and $\mathbf{f}^{(1)} = [0, -2a]^{\mathrm{T}}$. The companion transformation is

$$x_1 = \tau^{-1} X_1, \quad x_2 = \tau^{-2} X_2, \tag{3.135}$$

where $\tau = t - t_* = e^s$. The companion system reads

$$X_1' = X_2 + X_1, \tag{3.136.a}$$
$$X_2' = 4\,{X_1}^3 + 2\,X_2 - 2\,X_3 a, \tag{3.136.b}$$
$$X_3' = X_3. \tag{3.136.c}$$

■

Singularity analysis and unstable manifolds

We can now perform a local analysis of the companion system around its fixed points. By construction, there are at least two of them. The first fixed point, $\mathbf{X}_0 = \mathbf{0}$, is the origin and the second one is $\mathbf{X}_* = (\boldsymbol{\alpha}, 0)$. Around the origin, the linear eigenvalues are $\mathrm{Spec}(D(\mathbf{F}(\mathbf{0})) = \{-\mathbf{p}, q\}$; that is, they are simply given by the exponents of the scale-invariant solution $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$ together with the exponent q characterizing the non-dominant part of the vector field.

The second fixed point is more interesting. We find $\mathrm{Spec}(D\mathbf{F}(\mathbf{X}_*)) = \boldsymbol{\rho}$. The linear eigenvalues of the fixed point $\mathbf{X}_*$ of the companion system are the Kovalevskaya exponents of the original system around the singular solution with the extra eigenvalue q due to the embedding. The unstable manifold of the fixed point $\mathbf{X}_*$ can be parametrized by

$$\begin{aligned} \mathbf{X}(s) &= \mathbf{P}_{\mathrm{u}}(C_k e^{\rho_k s}, \ldots, C_N e^{\rho_N s}), \qquad \Re(\rho_i) > 0,\ i > k-1, \\ &= \mathbf{X}_* + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}(s) e^{(\boldsymbol{\rho}^{\mathrm{u}}.\mathbf{i})s}, \end{aligned} \tag{3.137}$$

and $N - k + 1 = \dim(W_{\mathrm{u}}(\mathbf{X}_*))$. Since $\tau \to 0$ as $s \to -\infty$, we have, in terms of the original variables,

$$\mathbf{x}(t) = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \hat{\mathbf{c}}_{\mathbf{i}}(\log \tau) \tau^{(\boldsymbol{\rho}^{\mathrm{u}}.\mathbf{i})} \right), \quad \boldsymbol{\rho}^{\mathrm{u}} = (\rho_k, \ldots, \rho_N), \tag{3.138}$$

where $\hat{c}_{\mathbf{i},j} = c_{\mathbf{i},j},\ j = 1, \ldots, n$. That is, the local series on the unstable manifold of the companion system's fixed point is the Ψ-*series* of the original system for the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$. This series contains as many free arbitrary constants as the sum of the number of positive Kovalevskaya exponents and the free constant t_* corresponding to the arbitrary location of the singularity. Therefore, a solution with $(n-1)$ positive Kovalevskaya exponents is a local expansion of the general solution.

Every non-zero fixed point of the companion system corresponds to another possible balance $\{\boldsymbol{\beta}, \mathbf{p}\}$. If $\bar{\mathbf{X}}$ is a fixed point, then $x_i(t) = \bar{X}_i \tau^{p_i}$ is an asymptotic solution of the original system. That is, an exact solution of $\dot{\mathbf{x}} = \mathbf{f}^{(0)}$. If $\bar{\mathbf{X}} = \mathbf{0}$, the corresponding solution is not a balance, but the origin of the original system. However, the converse is not true and if $\{\boldsymbol{\beta}, \mathbf{q}\}$ is another balance with $\mathbf{q} \neq \mathbf{p}$, then $X_* = (\boldsymbol{\beta}, 0)$ is not a fixed point of the companion system associated with the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$. Therefore, one way to analyze locally all possible balances of a given system is to find all possible decompositions of the vector field $\mathbf{f}$ and the corresponding vectors $\mathbf{p}$. Each of these decompositions corresponds to a companion system. The local analysis of all non-vanishing fixed points of these companion systems provides the local analysis of all possible balances of the original system. We next consider the behavior of each companion system around the fixed point $\mathbf{X}_*$.

3.8.6 Convergence of local solutions

We have proved the existence of formal series around the movable singularities and we now establish their convergence. In general, the local series (3.111) are not convergent. This difficulty is similar to one arising in the problem of the convergence of normal form transformations around a fixed point. In this case, it is well-known that unless particular conditions on the eigenvalues are satisfied, the normalizing transformations are divergent (Bruno, 1989). However, we can consider series (3.111) with only ascending powers by setting all arbitrary constants associated with Kovalevskaya exponents with non-positive real parts to zero. Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and let $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$ be a balance with Kovalevskaya exponents $\hat{\boldsymbol{\rho}} = \{\rho_1, \rho_2, \dots, \rho_n\}$, where $\rho_1 = -1$, and $(\rho_2, \dots, \rho_n)$ are ordered according to their real parts. To this set, we add $\rho_{n+1} = q \in \mathbb{Q}^+$ to obtain $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_{n+1}\}$. Let $\boldsymbol{\rho}^{\mathrm{u}}, \boldsymbol{\rho}^{\mathrm{c}}$, and $\boldsymbol{\rho}^{\mathrm{s}}$ be the vectors of Kovalevskaya exponents with, respectively, strictly positive, vanishing and strictly negative real parts. Take the series obtained by considering only the Kovalevskaya exponents with positive real parts. The series obtained from (3.111) by setting all the arbitrary constants associated with $\boldsymbol{\rho}^{\mathrm{s}}, \boldsymbol{\rho}^{\mathrm{c}}$ to zero are

$$\mathbf{x}(t) = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{|\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}^{\mathrm{u}}(\log \tau) \tau^{\boldsymbol{\rho}^{\mathrm{u}} \cdot \mathbf{i}} \right), \tag{3.139}$$

where we used $\tau = t - t_*$ to introduce explicitly the arbitrary constant associated with the Kovalevskaya exponent $\rho_1 = -1$. We want to prove the convergence of the Ψ-series for $\tau = (t - t_*)$ and the arbitrary constants $\mathbf{C}^{\mathrm{u}} = (C_k, \dots, C_n)$ small enough. If a Ψ-series reduces to a Puiseux series (*i.e.*, with rational Kovalevskaya exponents and without logarithmic terms), its convergence has been proved by Adler and van Moerbeke (1989b) and Brenig and Goriely (1994). The work on singular analysis for PDEs by Kichenassamy and co-workers (1993a; 1993b; 1995) strongly suggest that that the Ψ-series are convergent in general. This has been successfully demonstrated on many specific examples (Sachdev & Ramanan, 1993; Hemmi & Melkonian, 1995; Melkonian & Zypchen, 1995; Abenda, 1997) and, in general, for planar vector fields by Delshams and Mir (1997). We now give a general proof of the convergence of Ψ-series (Goriely, 2001).

Theorem 3.3 *Let $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$ be a balance of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and let $\mathbf{x}(t, \mathbf{C}^{\mathrm{u}})$ be the local Ψ-series (3.139) around the singularity t_* with arbitrary coefficients $C_k, \dots, C_n$. Then, there exists $\epsilon \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $0 < \alpha < \Re(\rho_i)$, $i = k, \dots, n+1$ and $\delta \in \mathbb{R}$, such that for $|\mathbf{a}| < \delta$, $0 < |t - t_*| < \epsilon$ and $M \in \mathbb{R}$*

sufficiently large we have

$$|\mathbf{x}(t, \mathbf{a})| < M|\mathbf{a}|\tau^{\alpha}. \tag{3.140}$$

To prove this result, we use the companion transformation introduced in the previous section to map the local series (3.139) to the local series around the fixed point (3.137). The convergence of the Ψ-series reduces to the convergence of the exponential series for the companion system as $s \to -\infty$. This is guaranteed by Lemma 3.2.

Proof. The proof of this theorem relies on Lemma 3.2 applied to a family of companion systems. We know the local solution on the unstable manifold of the fixed point $\mathbf{X}_*$ to be such that $|\mathbf{X}(s, \mathbf{a})| < K|\mathbf{a}|e^{\alpha s}$ for some $s < s_1$ real. Now, to test the convergence of the Ψ-series for $t \in \mathbb{C}$, we follow the argument given by Delshams and Mir (1997) and consider the family of companion systems obtained by the transformation $\tau = e^s e^{i\theta}$ where $\theta \in [0, 2\pi)$. Taking s real and fixing θ implies that τ is the distance from the origin on a ray of angle θ with the positive real axis. For each fixed θ, we obtain a new companion system $\mathbf{F}_\theta$ for which Lemma 3.2 applies. That is, we find for each θ a value $K_\theta \in \mathbb{R}$. By taking $M = \max\{K_\theta, \theta \in [0, 2\pi)\} \in \mathbb{R}$, the Ψ-series converges for all τ in a punctured disk of radius ϵ. □

The case of a non-hyperbolic balance, when one or more eigenvalues have zero real parts, is more complicated since it leaves the possibility of having $\log(\tau)$ terms in the first term of the expansions, which clearly diverges as $\tau \to 0$. The convergence of the local series associated with the Kovalevskaya exponents with negative real parts cannot be established in the general case. Indeed, there is no arbitrary constant associated with the non-dominant exponent, q. Since the local series will contain both negative powers of τ associated with negative Kovalevskaya exponents and positive powers of τ associated with q, there is no ordering of the powers as $\mathbf{i}$ increases and convergence cannot be guaranteed. However, in the particular case $q = 0$, when the vector field is weight-homogeneous, we have the following result.

Proposition 3.4 *Let $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$ be a balance of a weight-homogeneous vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and let $\mathbf{x}(t, \mathbf{C}^{\mathrm{s}})$ be the local Ψ-series*

$$\mathbf{x}(t) = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{|\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}^{\mathrm{s}}(\log \tau)\tau^{\boldsymbol{\rho}^{\mathrm{s}}.\mathbf{i}} \right), \tag{3.141}$$

around the singularity t_* with arbitrary coefficients $C_1, \ldots, C_k$. Then, there exist $\epsilon \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\Re(\rho_i) < \alpha < 0$, $i = 1, \ldots, k$ and $\delta \in \mathbb{R}$ such that, for $|\mathbf{a}| < \delta$, $|t - t_*| > \epsilon$ and $M \in \mathbb{R}$ sufficiently large,

$$|\mathbf{x}(t, \mathbf{a})| < M|\mathbf{a}|\tau^{\alpha}. \tag{3.142}$$

Proof. The proof follows along the same lines as the proof of Theorem 3.3 where we we consider the transformation $\tau = e^{-s}e^{i\theta}$. Hence, the local solution parametrizing the unstable manifold of the point X_* for this new companion system corresponds to the local series (3.141). Therefore, the convergence of this series for $s \to -\infty$ guarantees the convergence of (3.141) for $|t - t_*| > \epsilon$. □

3.8.7 A short list of singularity analyses

A priori, the procedure to build the local solutions as outlined in the previous sections does not allow us to obtain global information such as the Painlevé property, the existence of first integrals, or special structures in phase space. The problem now is to show that these local series contain important information regarding the global behavior of the solutions. Depending on the type of information one is interested in, the conditions imposed on the local series will be different. The main formal procedure outlined here is standard but the conditions are different. Each procedure or test is designed to prove some relations between local singularity analysis and global integrability or nonintegrability. Accordingly, we distinguish different types of singularity analyses.

1. **The Painlevé tests:** The Painlevé tests are designed to prove the Painlevé property for a given system. Therefore, the key feature is to show that the local solutions are single-valued. The local analysis however only provides necessary conditions for the Painlevé property. The simpler test only requires that the scale-invariant solution be single-valued, that is, $\mathbf{p} \in \mathbb{Z}^n$. If we want to further rule out algebraic singularities, we also require that all Kovalevskaya exponents be integers (Painlevé test #1). We can consider now the local series with ascending powers by keeping only the positive Kovalevskaya exponents. To rule out logarithmic singularities in these solutions, we compute all the compatibility conditions up to the largest Kovalevskaya exponent and find the conditions for the absence of logarithmic coefficients (Painlevé test #2). Finally, we build iteratively the general formal local series (3.111) and check that for all $\mathbf{i}$, the coefficients $\mathbf{c_i}$ are constant (Painlevé test #3). That is, the local solutions are Laurent series. However, in general, this last step cannot be completed in a finite number of steps,since the resonance relation $\boldsymbol{\rho}.\mathbf{i} = \rho_j$ will hold for infinitely many $\mathbf{i}$, except in the particular case where $(n-1)$ Kovalevskaya exponents are positive integers.
2. **The weak-Painlevé test:** The weak-Painlevé test extends the Painlevé test by considering certain types of algebraic singularities in the hope to find necessary conditions for algebraic integrability. It is usually related to integrable two-degree-of-freedom Hamiltonian systems and is discussed in detail in Chapter 4 where variable transformations are introduced to map systems with the weak-Painlevé property to systems with the Painlevé property.
3. **Yoshida's method for homogeneous vector fields:** Yoshida's analysis of weight-homogeneous systems provides necessary conditions for the non-existence of algebraic first integrals and is solely based on the rationality of Kovalevskaya exponents. As an extension of this test, we will also consider the obstruction to algebraic integrability provided by the presence of logarithmic terms (both methods are discussed in Chapter 5).
4. **Ψ-series analysis:** The Ψ*-series* are the general series containing logarithmic terms. It will be shown in Chapter 5 that the existence of these series rules out complete integrability. However, they still contain some information on the clustering of singularities in complex time and the splitting of homoclinic orbits in phase space (see Chapter 7).

3.9 The Painlevé tests

The Painlevé property is a global property for differential equations. There is no decision procedure (that is, no algorithmic procedure) to decide if an equation has the Painlevé property. A *Painlevé test* is an algorithmic test dedicated to build necessary conditions for the Painlevé property. We now describe some of the Painlevé tests, in order of increasing complexity, and discuss their sufficiency with respect to the Painlevé property. In recent history, the Painlevé test for ODEs was first shown to be useful to find integrable cases for the Lorenz system (Segur, 1982) and then, many other physical systems were studied and new cases of integrability unraveled (Menyuk *et al.*, 1982; Menyuk *et al.*, 1983; Ramani *et al.*, 1989).

3.9.1 Painlevé test #1: The Hoyer-Kovalevskaya method

Historical digression II

The idea that the analysis of the local solutions around complex time singularities could lead to some insights on the integrability of a nonlinear systems of ordinary differential equations was first advanced by Paul Hoyer in 1879. Paul Hoyer was a student of Weierstrass and, studied in his doctoral thesis the integrability of the system (Hoyer, 1879)

$$\dot{x}_1 = a_1 x_2 x_3 + a_2 x_3 x_1 + a_3 x_1 x_2, \tag{3.143}$$

$$\dot{x}_2 = b_1 x_2 x_3 + b_2 x_3 x_1 + b_3 x_1 x_2, \tag{3.144}$$

$$\dot{x}_3 = c_1 x_2 x_3 + c_2 x_3 x_1 + c_3 x_1 x_2. \tag{3.145}$$

The basis of his analysis was to look for conditions on the parameters such that the Kovalevskaya exponents are integer or rational numbers. This provides necessary conditions for the local solutions around the singularities to be either Laurent or Puiseux series. He also considered cases where the solution can be expanded in Puiseux series and found exact solutions (mostly by using clever ansatz).

Before her work on the Euler equations, S. Kovalevskaya also studied the integrability of simple quadratic systems of ODEs. Her unpublished work appeared in her collected letters and was only recently brought to the general attention (Grammaticos *et al.*, 1990a). Along the lines of Hoyer, she studied in 1884 the integrability of the three-dimensional Lotka-Volterra system (obviously, before the work of Lotka and Volterra in the 1920's!)

$$\dot{x}_1 = x_1(a_1 x_1 + a_2 x_2 + a_3 x_3), \tag{3.146}$$

$$\dot{x}_2 = x_2(b_1 x_1 + b_2 x_2 + b_3 x_3), \tag{3.147}$$

$$\dot{x}_3 = x_3(c_1 x_1 + c_2 x_2 + c_3 x_3). \tag{3.148}$$

Soon after, Kovalevskaya worked on the problem of rigid body motion and obtained her seminal result already mentioned in Section 2.1.1. Her work was twofold. First, she found a new case for which she built the general solution in terms of theta functions of two variables. Second, she showed that apart from the four known special cases, there is no other case for which the solution is single-valued (Kowalevski, 1889a; Kowalevski, 1889b; Tabor, 1984). The Euler equations (2.16) read

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{J}\boldsymbol{\omega}) = \mathbf{X} \times \boldsymbol{\gamma}, \tag{3.149.a}$$

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0, \tag{3.149.b}$$

where $\boldsymbol{\omega}$ is the angular velocity and the $\boldsymbol{\gamma}$ describes the orientation of the top. Kovalevskaya began by noticing that in Euler and Lagrange cases, the six variables $(\boldsymbol{\omega}, \boldsymbol{\gamma})$ are single-valued functions of time whose only singularities in the complex t-plane are poles. The general solution has the property that it can be expanded in Laurent series with finite principal parts. She then asked whether there are, for general values of the parameters, Laurent expansions with five free parameters (in order to describe the general solution) of the form

$$\boldsymbol{\omega} = t^{\mathbf{p}} \sum_{i=0}^{\infty} \mathbf{a}_i t^i, \qquad \boldsymbol{\gamma} = t^{\mathbf{q}} \sum_{i=0}^{\infty} \mathbf{b}_i t^i, \tag{3.150}$$

with $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^3$, $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{C}^3$. In order for these series to represent the general solution, the following conditions must be fulfilled:

1. The series is a formal solution and is convergent in a punctured disk
2. Five coefficients should be left undetermined.

To show the formal existence of the series for this system, Kovalevskaya computed the scale-invariant solutions and for each of these solutions the matrix K. Then, she looked for conditions on the parameters for the eigenvalues of K (the Kovalevskaya exponents) to be integers (Kowalevski, 1889a):

> *Afin que les séries ... contiennent le nombre suffisant de constantes arbitraires, il faut que le déterminant de ces équations linéaires ... s'évanouisse pour cinq valeurs différentes de m égales à des nombres entiers positifs.*[11]

Kovalevskaya found that the only cases for which these conditions are fulfilled are the three classical cases and a new one for which $J_1 = J_2 = 2J_3$ and $X_3 = 0$ with the fourth first integral

$$I_4 = \left[(\omega_1 + i\omega_2)^2 + X_1(\gamma_1 + i\gamma_2)\right] \left[(\omega_1 - i\omega_2)^2 + X_1(\gamma_1 - i\gamma_2)\right]. \tag{3.151}$$

However, this was not enough for Kovalevskaya and she pursued her task by integrating the equations of motions in terms of theta function of two variables and hyperelliptic integrals (the two objects are related by the Jacobi inversion procedure).

Kovalevskaya was vividly criticized by the Russian academician Markov on a minor point related to the non-existence of other cases where the equations could be integrated. This problem was settled by Lyapunov, who proved in 1894 that Kovalevskaya's claim was correct (Cooke, 1984, p. 118). We reproduce in the next section the argument of Lyapunov to prove the single-valuedness of the integrable cases of the Euler top.

The Kovalevskaya-Hoyer procedure

The procedure used by Kovalevskaya and Hoyer to test the single-valuedness of the general solution rests on the computation of the Kovalevskaya matrix.

Theorem 3.4 (Painlevé test #1) *Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{f}(\mathbf{x})$ is analytic and assume that it has the Painlevé property. Then, for all possible balances $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$, the dominant exponents are integer valued, the Kovalevskaya matrix K is semi-simple and the Kovalevskaya exponents $\boldsymbol{\rho}$ are integer valued.*

To prove this theorem, we need two versions of Painlevé's α-lemma.

Lemma 3.3 *Let $\{\boldsymbol{\alpha}, \mathbf{p}\}$ be a dominant balance for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. If $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ enjoys the Painlevé property then so does $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$.*

[11] "In order for the series to contain enough arbitrary constants, it is necessary that the determinant of these linear equations vanishes identically for five different values of m equal to positive integers."

Proof. Apply the scale transformation $\mathbf{x} \to \epsilon^p \mathbf{x},\ t \to \epsilon t$ to the vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ to obtain

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \epsilon) = \sum_{i=0}^{m} \epsilon^{q^{(i)}} \mathbf{f}^{(i)}(\mathbf{x}), \tag{3.152}$$

where $q^{(0)} = 0$ and $q^{(i)} \in \mathbb{Q}^+ \ \forall\ i > 0$ as defined by (3.110). The original system has the Painlevé property and, as a consequence, its general solution is single-valued. Let $\mathbf{X}(t)$ be the local expansion of such a solution around the singularity characterized by the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$. Then, system (3.152) has a single-valued solution, $\mathbf{X}(t, \epsilon)$, analytic in ϵ such that $\mathbf{X}(t, \epsilon = 1) = \mathbf{X}(t)$. We can now apply Painlevé's α-lemma (Theorem 3.1) and conclude that the solution $\mathbf{X}(t, \epsilon = 0)$ of $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ is also single-valued. □

Lemma 3.4 *Let $\{\boldsymbol{\alpha}, \mathbf{p}\}$ be a dominant balance for the weight-homogeneous system $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$. If $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ enjoys the Painlevé property then the fundamental solution of $t\dot{\mathbf{u}} = K\mathbf{u}$ with $K = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p})$ is single-valued.*

Proof. Consider the solution $\mathbf{x} = \sum_{i=0}^{\infty} \mathbf{x}^{(i)} \epsilon^i$ where $\mathbf{x}^{(0)} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$ and ϵ is an arbitrary parameter. Note that $\mathbf{f}^{(0)}$ does not depend on ϵ. The equation for $\mathbf{x}^{(1)}$ reads

$$t\dot{\mathbf{x}}^{(1)} = K\mathbf{x}^{(1)}, \tag{3.153}$$

where $K = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p})$. According to Painlevé's α-lemma, if $\mathbf{x}(t)$ is single-valued, then $\mathbf{x}^{(i)}$ is single-valued for all i. Hence we conclude that the solutions of $t\dot{\mathbf{u}} = K\mathbf{u}$ are single-valued. □

Proof. (Theorem) For a given balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$, we apply Lemma 3.3 to the system $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ and show that non-integer Kovalevskaya exponents or dominant exponents $\mathbf{p}$ imply that the solution is not single-valued. First, the solution $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$ is an exact particular scale-invariant solution of $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ obtained by setting all of the arbitrary constants to zero. If the general solution is single-valued, then all particular solutions are also single-valued. Hence, we have $\mathbf{p} \in \mathbb{Z}^n$. Consider now the variational equation around the solution $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$:

$$t\dot{\mathbf{u}} = K\mathbf{u}, \tag{3.154}$$

where $K = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p})$. If $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ has a single-valued general solution then the solution of the variational equation around any particular solution must also be single-valued (Lemma 3.4). By definition, the Kovalevskaya exponents are the Fuchs indices of the linear system (3.154). Therefore, the general solution $\mathbf{u} = t^K\mathbf{C}$ of (3.154) is single-valued if and only if K is semi-simple and $\mathrm{Spec}(K) \in \mathbb{Z}^n$ (see Proposition 3.1). □

Example 3.15 The Hénon-Heiles Hamiltonian. Hamilton's equations for the Hénon-Heiles Hamiltonian (Hénon & Heiles, 1964; Weiss, 1984; Tabor, 1989)

$$H = \frac{1}{2}(x_3^2 + x_4^2 + ax_1^2 + x_2^2) + bx_1^2x_2 - \frac{1}{3}x_2^3, \tag{3.155}$$

are

$$\dot{x}_1 = x_3, \tag{3.156.a}$$
$$\dot{x}_2 = x_4, \tag{3.156.b}$$
$$\dot{x}_3 = -ax_1 - 2bx_1x_2, \tag{3.156.c}$$
$$\dot{x}_4 = -x_2 - bx_1^2 + x_2^2. \tag{3.156.d}$$

We look for conditions on the parameters (a, b) such that the system passes Painlevé test #1. That is, we check that the exponents $\mathbf{p}$ are integers and that the Kovalevskaya matrix is semi-simple with integer eigenvalues for all possible balances. There is a unique weight-homogeneous decomposition of the vector field provided by the exponents $\mathbf{p} = (-2, -2, -3, -3)$ which is

$$\mathbf{f}^{(0)} = \begin{bmatrix} x_3 \\ x_4 \\ -2bx_1x_2 \\ -bx_1^2 + x_2^2 \end{bmatrix}, \quad \mathbf{f}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ -ax_1 \\ -x_2 \end{bmatrix}. \tag{3.157}$$

The non-dominant exponent associated with $\mathbf{f}^{(1)}$ is $q^{(1)} = 1$. The solutions of $\mathbf{p}\boldsymbol{\alpha} = \mathbf{f}^{(0)}(\boldsymbol{\alpha})$ lead to three different balances:

$$\boldsymbol{\alpha}^{(1)} = (0, 6, 0, -12), \tag{3.158.a}$$

$$\boldsymbol{\alpha}^{(2,3)} = (\frac{\pm 3}{b}\sqrt{b\,(2\,b-1)}, \frac{3}{b}, \frac{\pm 6}{b}\sqrt{b\,(2\,b-1)}, -\frac{6}{b}). \tag{3.158.b}$$

For each balance, we compute the eigenvalues of K, where

$$K = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 2b\alpha_2 & 2b\alpha_1 & 3 & 0 \\ 2b\alpha_1 & 2\alpha_2 & 0 & 3 \end{bmatrix}, \tag{3.159}$$

and obtain two different sets of Kovalevskaya exponents:

$$R^{(1)} = \left\{-1, 6, \frac{1}{2}(5 \pm \sqrt{1+48b})\right\}, \tag{3.160.a}$$

$$R^{(2,3)} = \left\{-1, 6, \frac{1}{2}(5 \pm \sqrt{-23+24/b})\right\}. \tag{3.160.b}$$

The condition that all Kovalevskaya exponents must be integers leads to three different cases:

$$\begin{aligned} b = 1, \quad & R^{(1)} = \{-1, -1, 6, 6\}, \\ & R^{(2,3)} = \{-1, 2, 3, 6\}, && (3.161.a) \\ b = 1/2, \quad & R^{(1)} = R^{(2,3)} = \{-1, 0, 5, 6\}, && (3.161.b) \\ b = 1/6, \quad & R^{(1)} = \{-1, 1, 4, 6\}, \\ & R^{(2,3)} = \{-1, -3, 6, 8\}. && (3.161.c) \end{aligned}$$

We conclude that unless $b \in \{1, 1/2, 1/6\}$, the system does not have the Painlevé property. At this stage there is no condition on the parameter a since it only appears in the non-dominant part of the vector field $\mathbf{f}^{(1)}$ and the Kovalevskaya exponents and dominant exponents only depend on the dominant part of the vector field. ■

Example 3.16 The Euler equations revisited. We consider the Euler equation (3.149) for rigid body motion and write them as a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ by setting $\mathbf{x} = (\boldsymbol{\omega}, \boldsymbol{\gamma})$. The explicit form of the vector field is given by equations (2.17). We look for conditions on the parameters $(\mathbf{J}, \mathbf{X}) \in \mathbb{R}^6$ such that the solutions are single-valued. The Euler system is weight-homogeneous with weight

$\mathbf{p} = (-1,-1,-1,-2,-2,-2)$. There are numerous balances given by the solution of the equation $\mathbf{p}\boldsymbol{\alpha} = \mathbf{f}(\boldsymbol{\alpha})$ depending on the values of the parameters. The Kovalevskaya exponents are given by the eigenvalues of the matrix

$$K = \begin{bmatrix} 1 & \frac{(J_2-J_3)\alpha_3}{J_1} & \frac{(J_2-J_3)\alpha_2}{J_1} & 0 & -\frac{X_3}{J_1} & \frac{X_2}{J_1} \\ \frac{(J_3-J_1)\alpha_3}{J_2} & 1 & \frac{(J_3-J_1)\alpha_1}{J_2} & \frac{X_3}{J_2} & 0 & -\frac{X_1}{J_2} \\ \frac{(J_1-J_2)\alpha_2}{J_3} & \frac{(J_1-J_2)\alpha_1}{J_3} & 1 & -\frac{X_2}{J_3} & \frac{X_1}{J_3} & 0 \\ 0 & -\alpha_6 & \alpha_5 & 2 & \alpha_3 & -\alpha_2 \\ \alpha_6 & 0 & -\alpha_4 & -\alpha_3 & 2 & \alpha_1 \\ -\alpha_5 & \alpha_4 & 0 & \alpha_2 & -\alpha_1 & 2 \end{bmatrix}. \tag{3.162}$$

Following the analysis of Lyapunov (Lyapunov, 1894; Leimanis, 1965), we consider three different balances in order to find the values of the parameters for which the solution may have the Painlevé property. That is, according to Painlevé test #1, the values of the parameters are those for which the Kovalevskaya exponents are integers and the matrix K is semi-simple.

First assume, by contradiction, that J_1, J_2, J_3 are all different and, without loss of generality, that $J_1 > J_2 > J_3 > 0$. In this case, there is a simple solution for $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha}^{(1)} = \Big(\frac{\sqrt{J_2J_3}}{\sqrt{(J_3-J_1)(J_1-J_2)}}, \frac{\sqrt{J_1J_3}}{\sqrt{(J_2-J_3)(J_1-J_2)}}, \frac{-\sqrt{J_1J_2}}{\sqrt{(J_2-J_3)(J_3-J_1)}}, 0, 0, 0\Big). \tag{3.163}$$

For the balance under consideration, $R^{(1)} = \{-1, 1, 2, 2, 2, 3\}$ and the Kovalevskaya exponents are all integers. However, in general, there are only two eigenvectors for the eigenvalue $\rho = 2$ and the matrix K is not semi-simple. For K to be semi-simple, we check that the minors of order 4 and 5 of the matrix $(K - 2\mathrm{I})$ vanish identically. This leads to the following condition on the parameters:

$$X_1\sqrt{J_1(J_2-J_3)} + X_2\sqrt{J_2(J_3-J_1)} + X_3\sqrt{J_3(J_1-J_2)} = 0. \tag{3.164}$$

Since $J_1 > J_2 > J_3$, the second term is imaginary unless $X_2 = 0$. This leads to the Hess-Appelrot case:

$$X_2 = 0, \text{ and } \sqrt{J_1(J_2-J_3)}X_1 + \sqrt{J_3(J_1-J_2)}X_3 = 0, \tag{3.165}$$

where there exists a second integral (given by (2.70)). Now, to prove that the solution is multi-valued in the Hess-Appelrot case, we study another balance which exists when $X_2 = 0$ and at least one of the other parameters X_1 or X_3 does not vanish. This balance is given by

$$\boldsymbol{\alpha}^{(2)} = \Big(0, 2i, 0, \frac{-2J_2}{iX_1 + X_3}, 0, \frac{-2iJ_2}{iX_1 + X_3}\Big). \tag{3.166}$$

For this balance, the Kovalevskaya exponents are $R^{(2)} = \{-1, 2, 3, 4, \rho_+, \rho_-\}$ where $\rho_\pm$ are the roots of

$$\rho^2 - \rho - \frac{2iX_1(J_1-2J_3)(J_2-J_3) + 2X_3(2J_2-J_3)(J_1-J_2)}{J_1J_3(iX_1+X_3)} = 0. \tag{3.167}$$

If $X_1X_3 \neq 0$, the Kovalevskaya exponents $\rho_\pm$ are not real (since $\rho_+\rho_-$ is not real) and the general solution is not single-valued. Assume that $X_1 = 0$ and $X_3 \neq 0$; from (3.165), we conclude that $J_2 = J_3$. This however is impossible since we have assumed that all of the J_i are different. Hence, we conclude that unless $X_1 = X_2 = X_3 = 0$, there is no single-valued solution with J_1, J_2, J_3 all different. The case $X_1 = X_2 = X_3 = 0$, is the Euler-Poinsot case where the center of gravity is the fixed point and is known to be completely integrable with single-valued solution.

Now, we can consider the case where two of the parameters J_1, J_2, J_3 are equal, say $J_1 = J_2$. Then, by a rotation around the x_3 axis, we can always choose $X_2 = 0$. From the previous analysis, we know that if $X_2 = 0$ there is a balance given by $\boldsymbol{\alpha}^{(2)}$. If $J_1 = J_3$, the last term of (3.167) simplifies to $-2iX_1(J_2 - J_3)/(J_3(iX_1 + X_3))$. That is, $\rho_\pm$ are real only if

1. $J_1 = J_2 = J_3$ (the completely symmetric case);
2. $X_1 = X_2 = 0$ and $J_1 = J_2$ (the Lagrange and Poisson case);
3. $X_3 = 0$, then $\rho_\pm = 1/2 \pm 1/2\sqrt{-7 + 8\lambda}$ which implies that $\lambda = B/C$ must be an integer. However, since $X_2 = X_3 = 0$, there is another balance similar to $\boldsymbol{\alpha}^{(2)}$ given by

$$\boldsymbol{\alpha}^{(3)} = (0, 0, 2i, \frac{-2J_3}{X_1}, \frac{-2iJ_3}{X_1}, 0), \tag{3.168}$$

   for which the Kovalevskaya exponents are $R^{(3)} = \{-1, 2, 3, 4, (2 - \lambda)/\lambda, 2(\lambda - 1)/\lambda\}$. These Kovalevskaya exponents are integers only if $\lambda = 1$ (the completely symmetric case) or $\lambda = 2$, the Kovalevskaya case ($J_1 = J_2 = 2J_3$ and $X_2 = X_3 = 0$). Finally, Kovalevskaya proved that this case can indeed be integrated exactly and that the general solution is single-valued.

The following theorem summarizes this analysis.

Theorem 3.5 (Kovalevskaya-Lyapunov) *For real parameters* $\mathbf{J}, \mathbf{C}$ *such that* J_1, J_2, J_3 *are all different from zero, the only cases where the Euler equation has a single-valued general solution are* (*i*) *the complete symmetric case* ($J_1 = J_2 = J_3$)*;* (*ii*) *the Euler-Poinsot case* ($X_1 = X_2 = X_3 = 0$)*,* (*iii*) *The Lagrange-Poisson case* ($X_1 = X_2 = 0$ *and* $J_1 = J_2$) *and* (*iv*) *The Kovalevskaya case* ($J_1 = J_2 = 2J_3$ *and* $X_3 = 0$)*.*

There exist however cases where particular solutions are single-valued. These solutions are obtained by specifying the value of some of the first integrals. For instance, the Hess-Appelrot case (3.165) (also known as the *loxodromic pendulum*) gives rise to exact single-valued solutions and there is a plethora of other cases of partial integrability (Leimanis, 1965, p. 91). ■

Remark. Clearly, the Kovalevskaya procedure to test single-valuedness is not sufficient. The Duffing system (3.92), for instance, has integer Kovalevskaya exponents $\{-1, 4\}$ but, in general, the compatibility conditions are not satisfied. Therefore, there exists a convergent solution in a punctured disk around a movable singularity which is not single-valued. It is therefore natural to add the requirement that every possible solution with ascending power be a Laurent series. This is the content of Painlevé test #2.

3.9.2 Painlevé test #2: The Gambier-ARS algorithm

The Painlevé test #2 is the central procedure to test the Painlevé property. The main procedure was first given by Gambier in 1910 (Conte, 1999) and rediscovered by Ablowitz, Ramani and Segur

(1980b). Nowadays, it is the most widely used test and it most useful for finding new integrable systems and for generalizing the method to PDEs and discrete systems. It rests on the following theorem.

Theorem 3.6 (Painlevé test #2) *Consider a system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *where* $\mathbf{f}(\mathbf{x})$ *is analytic and assume that it has the Painlevé property. Then, for all possible balances* $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$*, the dominant exponents* $\mathbf{p}$ *are integer valued, the Kovalevskaya matrix* K *is semi-simple, the Kovalevskaya exponents* $\boldsymbol{\rho}$ *are integer valued, and the local series*

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{i=1}^{\infty} \mathbf{a}_i \tau^i \right), \tag{3.169}$$

is a Laurent series with as many arbitrary constants as the number of strictly positive Kovalevskaya exponents.

Proof. From Theorem 3.4, we know that for each balance, $\mathbf{p} \in \mathbb{Z}^n$, and that K is semi-simple with integer eigenvalues. Therefore, we have to check that the single-valuedness of the solution given by the Painlevé property implies that the local series with ascending powers (*i.e.*, with positive Kovalevskaya exponents) are Laurent series. Consider the balance $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$ with Kovalevskaya exponents $\hat{\boldsymbol{\rho}} = (\rho_1, \ldots, \rho_n)$. Let $\rho^{\mathrm{u}} = (\rho_k, \ldots, \rho_n, \rho_{n+1} = q)$, where, as before, $1/q \in \mathbb{Z}^+$ is the least common denominator of the non-dominant exponents $q^{(1)}, \ldots, q^{(m)}$ associated with the weight-homogeneous decomposition of the vector field $\mathbf{f}$ with respect to $\mathbf{p}$. We know, from Theorems 3.2 and 3.3 that there exists a convergent local series of the form

$$\mathbf{x}(t) = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{|\mathbf{i}|=1}^{\infty} \mathbf{a}_{\mathbf{i}}(\log \tau) \tau^{\boldsymbol{\rho}^{\mathrm{u}}.\mathbf{i}} \right). \tag{3.170}$$

According to Theorem 3.4, $\hat{\boldsymbol{\rho}} \in \mathbb{Z}^n$. Therefore, this series can be written as a Laurent series with time-dependent coefficients

$$\mathbf{x}(\tau) = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{i=1}^{\infty} \mathbf{a}_i(\tau) \tau^i \right). \tag{3.171}$$

where some of the $\mathbf{a}_i(\tau)$ are, by contradiction, a function of $\log \tau$. Now, fix the arbitrary constant $\mathbf{C}^{\mathrm{u}} = (C_k, \ldots, C_n)$ and let $D_{\mathbf{C}}$ be a punctured disk centered at the origin where $\mathbf{x}(\tau)$ is convergent. Let γ be a closed path around $\tau = 0$ contained in $D_{\mathbf{C}}$. Consider a point t_1 on γ. Then, the local series (3.171) determines the analytic continuation of the solution, $\mathbf{x}(\tau)$, along γ. Clearly, one of the functions $\mathbf{x}(\tau)$ has a logarithmic branch point at $\tau = 0$ which is impossible. Since the functions $\mathbf{a}_i(\tau)$ are polynomial in $\mathbf{C}^{\mathrm{u}}$, as we vary $\mathbf{C}^{\mathrm{u}}$ in an open domain, we conclude that $\mathbf{a}_i$ must be constant for all i. That is, $\mathbf{x}(\tau)$ is a Laurent series. □

The algorithm

The method, known in the literature as the *ARS algorithm*, gives necessary conditions for the absence of logarithmic or algebraic branch points. It is divided into three steps. The first two steps are the conditions given by the Painlevé test #1 for the absence of algebraic branch points, the third checks for the absence of logarithmic branch points.

1. Find all possible balances $\mathcal{F}$. At this stage, the requirement for the absence of branch point is $p \in \mathbb{Z}^n$ for all $\mathcal{F} \in \widetilde{\mathcal{F}}$.

2. For each balance $\mathcal{F}$, compute the Kovalevskaya matrix

$$K = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p}). \tag{3.172}$$

   Check that (i) $\mathrm{Spec}(K) \in \mathbb{Z}^n$ and (ii) K is semi-simple.

3. For each balance and each positive Kovalevskaya exponent, compute the compatibility conditions and check that they vanish identically. That is, check the existence a formal Laurent series with as many arbitrary parameters as the number of positive Kovalevskaya exponents. If there are no vanishing Kovalevskaya exponents, the series is convergent.

Conditions (1) and $(2i)$ ensure that the singularity is not an *algebraic branch point* whereas conditions $(2ii)$ and (3) ensure it is not a *logarithmic branch point*. The case of a Laurent series with vanishing Kovalevskaya exponents is not covered by Theorem 3.6 but will be treated in the next section. However, since the existence of a Laurent series with positive and/or vanishing Kovalevskaya exponents can be tested in a finite procedure, we include it in the general algorithm and defer its proof until the next section. This test could simply be rephrased as: *Check that all local series with ascending powers are Laurent series.*

This test is a finite algorithmic procedure since there are finitely many balances and for each balance finitely many compatibility conditions (up to the last positive Kovalevskaya exponent). Once the last compatibility condition has been checked, we know, by construction, that there exists a formal Laurent series.

Example 3.17 Most of the examples of Section 3.8 were examples of systems passing Painlevé test #2. For instance, we looked for the existence of Laurent series for the three wave system (3.58) and found conditions on the parameters for the absence of logarithmic terms. The Duffing system (3.92) was found to have the Painlevé property for all cases where it passes Painlevé test #2. The Euler equation (3.149) also was found to have the Painlevé property when the local solutions with ascending powers are Laurent series. In the case of the Hénon-Heiles system (3.156), we found values of the parameters b where the system has integer Kovalevskaya exponents. However, the system does not have the Painlevé property for all values of a and conditions on a for the local series to be Laurent series have to be derived (see Exercises).

■

Example 3.18 The Chazy equation. The modified Chazy equation reads (Chazy, 1911; Chakravarty *et al.*, 1990; Fordy & Pickering, 1991)

$$\dddot{x} = -(16+8a)\dot{x}^2 + 2(7+2a)x\ddot{x}. \tag{3.173}$$

Written as a system with $\mathbf{x} = (x, \dot{x}, \ddot{x})$, the Chazy equation is weight-homogeneous with dominant exponents $\mathbf{p} = (-1, -2, -3)$ and has a unique balance $\boldsymbol{\alpha} = (-1/2, 1/2, -1)$. The Kovalevskaya exponents are readily found to be

$$\mathcal{R} = \{-1, -a - \sqrt{a^2-6}, -a + \sqrt{a^2-6}\}. \tag{3.174}$$

Since $\rho_2\rho_3 = 6$, the only values of a that lead to integer Kovalevskaya exponents are $a = -7/2$ with $\mathcal{R} = \{-1, 1, 6\}$; $a = -5/2$ with $\mathcal{R} = \{-1, 2, 3\}$; $a = 7/2$ with $\mathcal{R} = \{-1, -1, -6\}$ and $a = 5/2$ with $\mathcal{R} = \{-1, -2, -3\}$.

For $a = -7/2$, the equation reduces to $\dddot{x} = 12\dot{x}^2$. It is straightforward to check that the local expansion around $\tau = 0$ is a Laurent series, $x = t^{-1}\left(-1/2 + a_1 t + a_6 t^6 + \dots\right)$, and the equation can be directly integrated in terms of elliptic functions.

For $a = -5/2$, since the Kovalevskaya exponents are independent over the positive integers ($\rho_2 = 2$ and $\rho_3 = 3$ are relatively prime), the local expansion is a Laurent series (see Lemma 4.4).

For $a = 5/2$ and $a = 7/2$, the Kovalevskaya exponents are all negative. It turns out that the case $a = 5/2$ has the Painlevé property whereas the case $a = 7/2$ does not. Therefore, Painlevé test #2 fails to provide conditions to distinguish between these two cases and thus further refinements of the test are necessary. In the original test proposed by Ablowitz, Ramani, and Segur (1980a) and detailed in the review by Ramani, Grammaticos, and Bountis (1989), at least one of the balances should have $(n-1)$ positive Kovalevskaya exponents. However, this requirement is not necessary since the original Chazy equation (the one obtained for $a = 5/2$) has no balance with two positive Kovalevskaya exponents. However, Chazy demonstrated that this equation has the Painlevé property (for a modern demonstration of this property see Ablowitz and Clarkson (1991, p. 337)). Remarkably, despite the fact that the solution of the Chazy equation has a movable natural boundary, the solution is single-valued inside and outside the boundary.

■

It could be argued that in order to test the Painlevé property for the Chazy equation we could build Laurent series with descending powers $x = \tau^{-1}\sum_{i=1}^{\infty} a_i \tau^{-i}$. Since the system is weight-homogeneous, such series are convergent in an annulus around t_* and a test similar to Painlevé test #2 can be designed. However, if the system is not weight-homogeneous, these series are not well-defined and series with both ascending and descending powers are required to show the absence of branch points. The following example shows that even for weight-homogeneous system, Painlevé test #2 fails to provide the necessary conditions for the Painlevé property.

Example 3.19 Resonances between Kovalevskaya exponents. Consider the fourth-order equation

$$\frac{d^4x}{dt^4} - x\frac{d^3x}{dt^3} + 2\frac{dx}{dt}\frac{d^2x}{dt^2} = 0. \tag{3.175}$$

The corresponding system with $\mathbf{x} = (x, \dot{x}, \ddot{x}, \dddot{x})$ is weight-homogeneous with dominant exponents $\mathbf{p} = (-1, -2, -3, -4)$. There is a unique dominant balance given by $\boldsymbol{\alpha} = (-12, 12, -24, 72)$ and the Kovalevskaya exponents are $\mathcal{R} = \{-1, -2, -3, 4\}$. Painlevé test #2 is trivially satisfied since there is a unique positive Kovalevskaya exponent and no non-dominant exponent. Similarly, there is a Laurent series with descending powers. However, as we will show in Example 3.21, this system does not have the Painlevé property due to a resonance between the positive and negative Kovalevskaya exponents $-2 = 2\rho_3 + 1\rho_4$ which cannot be captured by looking at each Laurent series separately. ■

3.9.3 Painlevé test #3: The Painlevé-CFP algorithm

So far, in order to test the Painlevé property, we have considered the local series with ascending powers in τ and imposed that these k-parameter families of solutions are Laurent series. However, the general formal local solution is a series with both ascending and descending powers of τ given by

Theorem 3.2. It seems natural that in order for the solution to be single-valued, all local expansions must be single-valued, including the formal expansions. This is indeed the case, but the proof of Painlevé test #2 cannot be used since it relies on the convergence of Laurent series. In general, the formal expansions of the solutions are not convergent and we must use Painlevé's α-lemma to prove that the single-valuedness of the general solution implies that all local solutions are Laurent series. This refinement of the original Painlevé test was proposed by Conte, Fordy and Pickering (1993) and takes into account equations such as Chazy's and (3.175) which exhibit negative Kovalevskaya exponents (Fordy & Pickering, 1991; Conte, 1992).

Theorem 3.7 (Painlevé test #3) *Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{f}(\mathbf{x})$ is analytic and assume that it has the Painlevé property. Then, for all possible balances $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$, the dominant exponents, $\mathbf{p}$, are integer valued, the Kovalevskaya matrix K is semi-simple, the Kovalevskaya exponents, $\boldsymbol{\rho}$, are integer valued, and the formal expansion of the general solution*

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{i=-\infty}^{\infty} \mathbf{a}_i \tau^i \right), \tag{3.176}$$

is a Laurent series with $(n-1)$ arbitrary constants.

Note that the general Laurent series (3.176) contains only $(n-1)$ arbitrary parameters. The last arbitrary parameter is the position t_* of the movable singularity. In particular, if there exists a balance with $(n-1)$ positive Kovalevskaya exponents, then the expansion only contains ascending powers.

Proof. The basic idea of the proof is to start with the convergent Laurent series provided by Theorem 3.6. Let $\mathbf{x}^{(0)}(\tau)$ be such a solution and consider the general solution

$$\mathbf{x}(\tau) = \sum_{i=0}^{\infty} \mathbf{x}^{(i)}(\tau)\epsilon^i, \tag{3.177}$$

where ϵ is an arbitrary parameter that does not appear in the vector field. We can apply Theorem 3.1 to the vector field $\mathbf{f}(\mathbf{x})$ with respect to the parameter ϵ, and conclude that in order for $\mathbf{x}(\tau)$ to be single-valued, we need $\mathbf{x}^{(i)}$ to be single-valued for all $i > 0$. The series, $\mathbf{x}^{(i)}$, are solutions of the non-homogeneous linear equation

$$\dot{\mathbf{x}}^{(i)} = D\mathbf{f}(\mathbf{x}^{(0)})\mathbf{x}^{(i)} + \mathbf{y}^{(i)}, \tag{3.178}$$

where $\mathbf{y}^{(i)} = \mathbf{y}^{(i)}(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(i-1)})$ is obtained by Taylor expanding the vector $\mathbf{f}$ around the solution $\mathbf{x}^{(i)}$. Therefore, $\mathbf{y}^{(i)}$ is polynomial in its arguments. Moreover, $\tau = 0$ is a regular singular point (see Definition 3.1) for this equation. Therefore, by Proposition 3.1, there is a convergent Laurent series solution with constant coefficients of the form

$$\mathbf{x}^{(i)} = \tau^{\mathbf{p} - i\rho_{\min}} \left(\boldsymbol{\alpha}^{(i)} + \sum_{j=0}^{\infty} \mathbf{c}_j^{(i)} \tau^j \right), \tag{3.179}$$

where $\rho_{\min}$ is the smallest Kovalevskaya exponent (either $\rho_{\min} = -1$ or $\rho_{\min} = \rho_2$). This series contains $(n-1)$ arbitrary parameters provided by the homogeneous part of the equation. Since the general Laurent series (3.176) is uniquely defined around $\tau = 0$ (up to $(n-1)$ arbitrary constants), the sum (3.177) can be identified by formally rearranging the series in the limit $\epsilon \to 1$, with the formal Laurent series

$$\begin{aligned} \mathbf{x}(\tau) &= \sum_{i=0}^{\infty} \mathbf{x}^{(i)} \epsilon^i, \\ &= \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{i=-\infty}^{\infty} \mathbf{a}_i \tau^i \right). \end{aligned} \tag{3.180}$$

□

The algorithms

Based on the last theorem, there are two different ways to test the Painlevé property. In either method, we must show that series (3.176) is a Laurent series, (that is, it has constant coefficients). To this aim we must demonstrate the absence of logarithmic terms. The first algorithm consists of computing the coefficients of

$$\mathbf{x}(t) = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}} \tau^{\boldsymbol{\rho}.\mathbf{i}} \right), \qquad \boldsymbol{\rho}.\mathbf{i} = \sum_{j=k}^{n+1} \rho_j i_j, \tag{3.181}$$

for increasing values $|\mathbf{i}|$ of the multi-index $\mathbf{i}$ and checking that they are constant. We consider a vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and apply to it the following algorithm.

Painlevé test #3: first algorithm

1. Find all possible balances $\mathcal{F}$. Check that $\mathbf{p} \in \mathbb{Z}^n$ for all $\mathcal{F} \in \widetilde{\mathcal{F}}$.

2. For each balance $\mathcal{F}$, compute the Kovalevskaya matrix

$$K = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \text{diag}(\mathbf{p}). \tag{3.182}$$

   Check that *(i)* $\text{Spec}(K) \in \mathbb{Z}^n$ and *(ii)* K is semi-simple.

3. For each balance, compute the series (3.181) for increasing values of $\mathbf{i}$ and check that the coefficients $\mathbf{c}_{\mathbf{i}}$ are constant. Following the construction of the local series given in the proof of Theorem 3.2, two cases should be distinguished. First, if $\boldsymbol{\rho}.\mathbf{i} \neq \rho_j \ \forall\ j \leq n$, then the coefficients $\mathbf{c}_{\mathbf{i}}$ are uniquely determined and no condition needs to be checked. Second, if there exists $j \leq n$ such that

$$\boldsymbol{\rho}.\mathbf{i} = \rho_j, \tag{3.183}$$

   then, the coefficient $\mathbf{c}_{\mathbf{i}}$ is not uniquely determined and there is a compatibility condition to be checked to ensure that $\mathbf{c}_{\mathbf{i}}$ is constant.

Relation (3.183) is a *resonance* relation between the Kovalevskaya exponents. The same type of relation is known to occur in the computation of normal forms of vector fields around their fixed points in phase space. In the next section, we will show that the computation of the Laurent series is indeed equivalent to a normal form computation. In general, there are infinitely many resonance relations between the Kovalevskaya exponents since $\boldsymbol{\rho} \in \mathbb{Z}^n$ and $\rho_1 = -1$. There are two cases where the number of resonances is finite. First, when all the Kovalevskaya exponents are strictly negative (this implies that the system is weight-homogeneous). Second, when all but one of the Kovalevskaya exponents are strictly positive. In this case, the only negative Kovalevskaya exponent is $\rho_1 = -1$ which needs not be tested since the arbitrary constant associated with $\rho = -1$ is the arbitrary position of the singularity. Therefore, in general, the algorithm given here is not a finite decision procedure and there is no a priori bound on the number $\mathbf{i}$ for which one compatibility condition may not be satisfied.

The second algorithm follows Painlevé's α-method and consists of computing the Laurent series $\mathbf{x}(\tau) = \sum_{i=0}^{\infty} \mathbf{x}^{(i)}\epsilon^i$ for each i. This amounts to solving, to each order, a system of linear non-homogenous equations around a regular singular point and checking that the coefficients of the Laurent series are constant (Conte *et al.*, 1993).

Painlevé test #3: second algorithm

1. Find all possible balances $\mathcal{F}$. Check that $\mathbf{p} \in \mathbb{Z}^n$ for all $\mathcal{F} \in \widetilde{\mathcal{F}}$.

2. For each balance $\mathcal{F}$, compute the Kovalevskaya matrix

$$K = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p}). \tag{3.184}$$

   Check that *(i)* $\mathrm{Spec}(K) \in \mathbb{Z}^n$ and *(ii)* K is semi-simple.

3. For each balance, expand the solution $\mathbf{x}(\tau) = \sum_{i=0}^{\infty} \mathbf{x}^{(i)}\epsilon^i$, compute the series $\mathbf{x}^{(0)}$ with ascending powers, and check that it is a Laurent series. That is, check all of the compatibility conditions up to the largest Kovalevskaya exponent.

4. For increasing values of i, compute the series

$$\mathbf{x}^{(i)} = \tau^{\mathbf{p}-i\rho_{\min}} \left(\boldsymbol{\alpha}^{(i)} + \sum_{j=0}^{\infty} \mathbf{c}_j^{(i)} \tau^j \right), \tag{3.185}$$

   and check that it is a Laurent series. That is, check all of the compatibility conditions up to the largest Kovalevskaya exponent.

Each step in (4) is finite. However, as before, there is no a priori bound on i to ensure that no logarithmic term will enter in the expansion. Again, unless all of the Kovalevskaya exponents are strictly positive or strictly negative this algorithm is not a finite procedure

Rather than using the algorithms presented in this section, we delay the applications of the Painlevé test #3 to the next section where a general algorithm for the Painlevé test in terms of normal forms is given.

3.9.4 Painlevé property and normal forms

As discussed in the previous sections, the computation of the formal local series and the condition for the absence of logarithmic terms in the series is similar to a normal form computation. We use the companion transformation introduced in Section 3.8.5 to show that local requirements for the Painlevé property are equivalent to conditions on the linearizability of the companion systems' fixed points.

Normal forms of analytic vector fields

Again, we consider a system

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}), \tag{3.186}$$

around the fixed point $\mathbf{y} = \mathbf{0}$. In section 3.8.5, we built local series based on the eigenvalues $\boldsymbol{\lambda}$ of the Jacobian matrix at the origin of the form

$$\mathbf{y}(t) = \mathbf{P}(C_1 e^{\lambda_1 t}, \ldots, C_n e^{\lambda_n t}), \tag{3.187}$$

where $\mathbf{P}$ is a vector of formal power series in the arguments with polynomial coefficients in t and $C_1, \ldots, C_n$ are arbitrary constants. A formal series is a *pure series* if all the coefficients are constant (*i.e.*, a pure series in exponentials).

To discuss normal forms, it is easier to think of systems of ODEs in terms of a vector field. Let

$$\delta = \sum_{i=1}^{n} \mathbf{g}_i(\mathbf{y}) \partial_{\mathbf{y}_i} \tag{3.188}$$

be the vector field associated with system (3.186) and let $\boldsymbol{\lambda}$ be the vector of linear eigenvalues around the origin. This vector field can be simplified by a local change of coordinates. That is, a near-identity transformation must be found. This transformation θ is a formal power series such that

$$\delta_\theta = \theta^{-1} \delta \theta \tag{3.189}$$

is as simple as possible. This usually means that the new vector field does not contain resonant terms as discussed below. The condition that θ be a near-identity transformation ensures that it is invertible and that the new vector field shares the same linear eigenvalues as the original system. That is, the linear component of δ is invariant under the transformation.

For a generic choice of $\boldsymbol{\lambda}$, the new vector field δ_θ can be chosen to be linear and the transformation θ is both analytic and unique. However, when linearization is not possible, three interesting cases can occur (see Ecalle and Vallet (1998)):

1. **Resonance.** There is a *resonance* between the linear eigenvalues if there exists at least one vector of positive integer $\mathbf{m}$ (with possibly one $m_i = -1$) such that $\boldsymbol{\lambda}.\mathbf{m} = 0$. Then, in general, there is no formal power series transformation θ that can linearize the vector field, and the simplest form of the new vector field contains only resonant monomials. That is, $[\delta_\theta, \delta_{\text{lin}}] = 0$. Moreover, if the Newton polygon (see Section 4.5) of $\boldsymbol{\lambda}$ in the complex plane does not contain the origin, then θ is analytic and δ_θ is a polynomial vector field (Arnold, 1988a). If the linear part of the vector field is diagonal then a monomial $\mathbf{x}^{\mathbf{m}}$ is resonant for the jth equation if $\lambda_j = (\mathbf{m}, \boldsymbol{\lambda})$.

2. **Quasi-resonance.** Let $\boldsymbol{\omega}(\mathbf{m}) = \boldsymbol{\lambda}.\mathbf{m} \neq 0$. If there exists an increasing sequence $\{\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \ldots\}$ such that

$$\boldsymbol{\omega}(\mathbf{m}^{(i)}) \underset{i \to \infty}{\to} 0, \tag{3.190}$$

"fast enough", then either the vector field can be linearized by a divergent series, or analytically transformed to a vector field that contains all the *quasi-resonant* monomial terms (*i.e.*, monomial of the form $\mathbf{x}^{\mathbf{m}^{(i)}}$). The condition that the sequence converges "fast enough" can be given explicitly in terms of Diophantine relations on the sequence – see Bruno (1989). This situation, of great theoretical interest, can be quite complex to study especially since multiple sequences can have such a property.

3. **Nihilence.** We say that the system exhibits nihilence, if there exists at least one formal first integral $I = I(\mathbf{y})$, that is, a formal power series such that

$$\delta I = 0. \tag{3.191}$$

In this case, the existence of a first integral provides additional structure to the series and, among others, implies that the linear eigenvalues are resonant (see Theorem 5.2).

There is a natural relationship between local series around a fixed point and linearizable vector fields. Let $\delta = \mathbf{f}(\mathbf{y})\partial_{\mathbf{y}}$ be the vector field associated with the system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$. This vector field can be split, in a chart independent manner, into two parts (Ecalle, 1987)

$$\delta = \delta_{\text{diag}} + \delta_{\text{nil}}, \tag{3.192}$$

where δ_{nil} commutes with δ_{diag}:

$$[\delta_{\text{diag}}, \delta_{\text{nil}}] = 0, \tag{3.193}$$

and δ_{diag} can be linearized by a formal near-identity transformation

$$\theta^{-1}\delta_{\text{diag}}\theta = \delta_{\text{lin}}, \tag{3.194}$$

where δ_{lin} is the linearized vector field around the origin. A vector field is *linearizable* if and only if its *nilpotent* component δ_{nil} vanishes identically. A linear vector field $\delta_{rmlin} = A\mathbf{x}.\partial_{\mathbf{x}}$ is *semi-simple* if the matrix A is semi-simple.

Lemma 3.5 *The system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$ has pure local series around $\mathbf{y} = \mathbf{0}$ if and only if the vector field $\delta_{\mathbf{g}}$ is formally linearizable and the linear part δ_{lin} is semi-simple.*

Proof. First, assume that δ is formally linearizable and that $D\mathbf{g}(\mathbf{0})$ is semi-simple. Without loss of generality, we write $D\mathbf{g}(\mathbf{0})$ =diag($\boldsymbol{\lambda}$). There exists a near-identity formal change of variable $\mathbf{y} = \boldsymbol{\theta}(\mathbf{z})$ that linearizes the system. In the new variables, the vector field reads $\dot{\mathbf{z}}$ =diag($\boldsymbol{\lambda}$)$\mathbf{z}$. Hence, we have $z_i = C_i e^{\lambda_i t}$ for $i = 1, \dots, n$ and the substitution of this solution in the series $\mathbf{y} = \boldsymbol{\theta}(\mathbf{z})$ defines a pure series.

Second, assume that the system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$ has pure local series and, without loss of generality, that the Jacobian $D\mathbf{g}(\mathbf{0})$ is in his Jordan normal form. If the Jordan normal form is not diagonal, then the first term of the local series contains polynomials in t (Perko, 1996). Since the first terms of these series are the series solutions of the linearized system, $D\mathbf{g}(\mathbf{0})$ is diagonal and δ_{lin} is semi-simple. Now, consider the pure series $\mathbf{y} = \mathbf{P}(C_1 e^{\lambda_1 t}, \dots, C_n e^{\lambda_n t})$ where $\mathbf{P}$ is a formal power series in its arguments with constant coefficients. Since $D\mathbf{g}(\mathbf{0})$ is diagonal, $\mathbf{P}$ is of the form $\mathbf{P}(.) = \mathbf{Id}(.) + \mathbf{Q}(.)$, where $\mathbf{Id}(.)$ is the identity and $\mathbf{Q}(.)$ is a formal power series with no linear terms. Now, let $z_i = C_i e^{\lambda_i t}$ for all i and consider the change of variable $\mathbf{y} = \mathbf{P}(\mathbf{z})$. This transformation is both an invertible transformation, since $\mathbf{P}$ is a near-identity transformation, and a formal change of variables that linearizes the vector field δ. □

One can also realize that if $\delta_{\rm nil}$ is not identically zero, the normal form in the variables $\mathbf{z}$ is not linear and at least one equation, say z_1, has at least one monomial resonant term $\dot{z}_1 = \lambda_1 z_1 + \mathbf{z}^{\mathbf{m}} + \ldots$ where $\mathbf{m}$ is such that $(\boldsymbol{\lambda}, \mathbf{m}) = 0$. The local series of this equation contains polynomial terms in t as does the local series in the original variables.

Hence, the absence of resonant terms in the normal forms implies that the local solution can be expressed only in terms of exponentials (a pure solution). Conversely, the presence of resonant terms (or a non semi-simple linear part) implies that the coefficients of the local solutions are polynomials in t. This observation is well known in physics where the presence of the so-called "secular terms" (polynomials in t) in the center manifold dynamics is known to be associated with resonances between the linear eigenvalues. However, the existence of resonance amongst the eigenvalues is not enough to create secular terms since $\delta_{\rm nil}$ may vanish identically.

The normal form analysis can be restricted on the unstable, stable and center manifolds. For instance, consider the unstable manifold and assume that the variables $\mathbf{y} = (\mathbf{y}^{\rm s}, \mathbf{y}^{\rm c}, \mathbf{y}^{\rm u})$ have been chosen such that $D\mathbf{g}(0) = \text{diag}(\mathrm{A}^{\rm s}, \mathrm{A}^{\rm c}, \mathrm{A}^{\rm u})$ where $\text{Spec}(A^{\rm s,c,u}) = \boldsymbol{\lambda}^{\rm s,c,u}$ (the set of eigenvalues with negative, zero and positive real parts). Then, a formal transformation $\mathbf{y} = \boldsymbol{\theta}(\mathbf{z})$ can be found such that the new vector field in the variables $\mathbf{z}$ reads

$$\delta_{\boldsymbol{\theta}} = \mathbf{g}^{\rm s}(\mathbf{z})\partial_{\mathbf{z}^{\rm s}} + \mathbf{g}^{\rm c}(\mathbf{z})\partial_{\mathbf{z}^{\rm c}} + \mathbf{g}^{\rm u}(\mathbf{z})\partial_{\mathbf{z}^{\rm u}}. \tag{3.195}$$

The restriction of $\mathbf{g}$ to the unstable manifold obtained by setting $\mathbf{z}^{\rm s} = 0$ and $\mathbf{z}^{\rm c} = 0$ gives a polynomial vector field in the variables $\mathbf{z}^{\rm u}$:

$$\delta^{\rm u} = \mathbf{g}^{\rm u}(\mathbf{z}^{\rm u})\partial_{\mathbf{z}^{\rm u}}. \tag{3.196}$$

Moreover, the restriction of the transformation $\boldsymbol{\theta}$ to the local unstable manifold given by $\mathbf{y} = \boldsymbol{\theta}(\mathbf{z}^{\rm s} = 0, \mathbf{z}^{\rm c} = 0, \mathbf{z}^{\rm u})$ is analytic. If system (3.196) is linear and $A^{\rm u}$ semi-simple, the solution is $z_i^{\rm u} = C_i e^{\lambda_i^{\rm u} t}$ and the analytic transformation defines a convergent pure series.

Painlevé property and linearizability

Consider a vector field $\mathbf{f}(\mathbf{x})$ together with a weight-homogeneous decomposition characterized by the dominant exponents $\mathbf{p}$ and non-dominant exponents $q^{(1)}, \ldots, q^{(m)}$. We associate a companion system $\mathbf{X}' = \mathbf{F}(\mathbf{X})$ obtained by the transformation (3.130) $\mathbf{x} = t^{\mathbf{p}}\mathbf{X}$, $\mathbf{X}_{n+1} = t^q$ and $t = e^s$ where $1/q$ is the least common denominator of $\{q^{(1)}, \ldots, q^{(m)}\}$. The local expansions in exponential series of the unstable manifolds around the non-vanishing fixed points were shown to correspond to the local solutions of the original system around its movable singularities.

The Painlevé property implies that all local series are Laurent series. The three Painlevé tests, given in the previous sections, can easily be recast in terms of normal forms around the fixed point of the companion system. First, Painlevé test #1 implies that the dominant exponent and the Kovalevskaya exponents are integer valued.

Proposition 3.5 (Painlevé test #1) *Assume that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has the Painlevé property. Then, all of its companion systems are such that*
(1) The linear eigenvalues of the companion system around $\mathbf{X_0}$ and $\mathbf{X}_*$ are integer valued, and
(2) The Jacobian matrix of the companion system at $\mathbf{X}_*$ is semi-simple.

Proof. This result is a direct consequence of Theorem 3.4 together with the identification of the linear eigenvalues at $\mathbf{X_0}$ and $\mathbf{X}_*$ with, respectively, the exponents $\mathbf{p}$ and the Kovalevskaya exponents. □

Painlevé test #2 implies that the local series with ascending powers are Laurent series. This test can be written in terms of the dynamics of the companion system.

Proposition 3.6 (Painlevé test #2) *Assume that* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *has the Painlevé property. Then, all its companion systems are such that*
(1) The linear eigenvalues of the companion system around $\mathbf{X_0}$ *and* $\mathbf{X}_*$ *are integer valued,*
(2) The Jacobian matrix of the companion system at $\mathbf{X}_*$ *is semi-simple, and*
(3) The unstable manifold of $\mathbf{X}_*$ *is analytically linearizable.*

Proof. By definition, a system passes Painlevé test #2 if all of the local series solutions are Laurent series. In particular, this implies that for each balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$, we have $\mathbf{p} \in \mathbb{Z}^n$, which implies that $q \in \mathbb{Z}$ and $\hat{\boldsymbol{\rho}} \in \mathbb{Z}^n$. To prove the rest of the proposition, we restrict the results of Lemma 3.5 on the unstable manifold. First, we assume that the unstable manifolds of $\mathbf{X}_*$ is linearizable and we let $\mathbf{Y} = \mathbf{X} - \mathbf{X}_*$. If an unstable manifold is linearizable then it is analytically linearizable since the eigenvalues are all in the *Poincaré domain* (*i.e.*, the Newton polygon of the eigenvalues does not contain the origin). Therefore, there exists an analytic change of variables, $\mathbf{Y} = \boldsymbol{\theta}^{\mathrm{u}}(\mathbf{Z}^{\mathrm{u}})$, such that the vector field for $\mathbf{Z}^{\mathrm{u}}$ is linear. If the Jacobian matrix of the companion system at $\mathbf{X}_0$ is semi-simple, the solution is $Z_i^{\mathrm{u}} = C_i e^{\rho_i^{\mathrm{u}} s}$. The substitution of this solution in the analytic transformation, $\mathbf{Y} = \boldsymbol{\theta}^{\mathrm{u}}(\mathbf{Z})$, defines a convergent pure series, which written in terms of the original variable reads

$$\mathbf{x}(t) = \boldsymbol{\alpha}\tau^{\mathbf{p}} \left(\mathbf{1} + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \mathbf{c_i} \tau^{\boldsymbol{\rho}^{\mathrm{u}}.\mathbf{i}} \right). \tag{3.197}$$

This series is a Laurent series if $\mathbf{p}$ and $\boldsymbol{\rho}^{\mathrm{u}}$ are integers. Hence, the linear eigenvalues of $\mathbf{X_0}$ and the positive linear eigenvalues of $\mathbf{X}_0$ are integer valued. If, however, the Jordan block of the Jacobian matrix associated with the positive eigenvalues of the companion system at $\mathbf{X}_0$ is not semi-simple, then the solutions for $\mathbf{Z^u}$ introduce polynomials in the variable s which after transformation introduce logarithmic terms in (3.197). □

The converse is also true. If the unstable manifold of $\mathbf{X}_*$ is analytically linearizable, then the local series around the singularities with ascending powers are Laurent series. Assume that the system passes Painlevé test #2. Then, the local series around the singularities are all of form (3.197) with integers $\mathbf{p}$ and $\boldsymbol{\rho}^{\mathrm{u}}$. This series defines the local series parametrizing the unstable manifold of the fixed point $\mathbf{X}_*$ for the companion system

$$\mathbf{X} = \mathbf{X}_* + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \mathbf{c_i} e^{(\boldsymbol{\rho}^{\mathrm{u}}.\mathbf{i})s}. \tag{3.198}$$

By contradiction, the fact that all the coefficients $\mathbf{c_i}$ are constant implies that the unstable manifold of $\mathbf{X}_*$ is linearizable and that the Jacobian matrix of the companion system at $\mathbf{X}_0$ is semi-simple.

The existence of an analytic change of variables linearizing the dynamics of the companion system on the unstable manifold implies (*i*) that the Laurent solutions of the original system with ascending powers are all convergent, a result already proved by Adler and van Moerbeke (1989b), and Brenig and Goriely (1994) and generalized by Theorem 3.3; and (*ii*) that locally around the scale-invariant solutions, there exist k analytic first integrals where k is the number of positive eigenvalues. For a principal balance, this implies that locally around the scale-invariant solution $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$, the system is completely analytically integrable (see Chapter 5 for a converse statement).

The Painlevé property actually imposes much stronger conditions on the local structure of solutions of the companion systems, as demonstrated by the Painlevé test #3.

Proposition 3.7 (Painlevé test #3) *Assume that* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *has the Painlevé property. Then, all of its companion systems are such that*
(1) The linear eigenvalues of of the companion system around $\mathbf{X_0}$ *and* $\mathbf{X}_*$ *are integer valued,*
(2) The Jacobian matrix of the companion system at $\mathbf{X}_*$ *is semi-simple,*
(3) The companion system is formally linearizable around $\mathbf{X}_*$*.*

Proof. Consider the formal solutions around the singularity t_*,

$$\mathbf{x}(t) = \tau^{\mathbf{p}}\mathbf{P}(C_1\tau^{\rho_1}, \ldots, C_{n+1}\tau^{\rho_{n+1}}), \tag{3.199}$$

where $\mathbf{P}$ is a formal power series in its arguments whose coefficients are polynomial in $\log(\tau)$. That is,

$$\mathbf{x}(t) = \tau^{\mathbf{p}}\left(\boldsymbol{\alpha} + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \hat{\mathbf{c}}_{\mathbf{i}}\tau^{\boldsymbol{\rho}.\mathbf{i}}\right), \qquad \boldsymbol{\rho}.\mathbf{i} = \sum_{j=1}^{n+1}\rho_j i_j. \tag{3.200}$$

As a consequence of Painlevé's test #3, a necessary condition for the Painlevé property is that the formal solution (3.200) is a Laurent series, that is, for $\hat{\mathbf{c}}_{\mathbf{i}}$ to be constant for all $\mathbf{i}$. The companion transformation maps (3.200) to the local series

$$\mathbf{X} = \mathbf{X}_* + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}} e^{\boldsymbol{\rho}.\mathbf{i}s}, \quad \boldsymbol{\rho}.\mathbf{i} = \sum_{j=1}^{N}\rho_j i_j. \tag{3.201}$$

The condition for the Painlevé property is for $\mathbf{X}(s)$ to be a pure series. Lemma 3.5, in turn implies that the companion system is formally linearizable and that the Jacobian matrix of the system at $\mathbf{X}_*$ is semi-simple. □

Yet another algorithm

Based on the last proposition, necessary conditions for the Painlevé property can be obtained by computing the normal forms of the companion system around $\mathbf{X}_*$.

Painlevé test #3: third algorithm

1. Find all possible balances $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$ with non-dominant exponent q.

2. For each balance, apply the companion transformation $\mathbf{x} \to \tau^{\mathbf{p}}\mathbf{X}$, $X_n + 1 = \tau^q$ and $\tau \to e^s$ to obtain $\mathbf{X}' = \mathbf{F}(\mathbf{X})$.

3. For each companion system, check that every non-zero fixed point is formally linearizable. That is, show that the local normal forms are linear.

The advantage of this algorithm is that it relies on the computation of normal forms, a subject of considerable studies in computer algebra. There are to date many general results and various excellent algorithms for the computation of normal forms (Mersman, 1970; Arnold, 1988a; Elphick *et al.*, 1987; Bruno, 1989; Walcher, 1991).

Example 3.20 The Rabinovich system. The three-dimensional system

$$\dot{x}_1 = -\omega x_2 - \nu_1 x_1 - x_2 x_3, \tag{3.202.a}$$
$$\dot{x}_2 = \omega x_1 - \nu_2 x_2 - x_1 x_3, \tag{3.202.b}$$
$$\dot{x}_3 = -\nu_3 x_3 - x_1 x_2. \tag{3.202.c}$$

is a model of interacting waves (Bountis *et al.*, 1984; Kruskal *et al.*, 1990). The variables $\mathbf{x}$ denote the amplitude of the waves and the parameters $\boldsymbol{\nu}$ represent damping coefficients. The parameter ω is related to the amplitude of the feeder wave. This system has a unique weight-homogeneous decomposition related to $\mathbf{p} = (-1, -1, -1)$ and

$$\mathbf{f}^{(0)} = \begin{bmatrix} -x_2 x_3 \\ -x_1 x_3 \\ -x_1 x_2 \end{bmatrix}, \quad \mathbf{f}^{(1)} = \begin{bmatrix} -\omega x_2 - \nu_1 x_1 \\ \omega x_1 - \nu_2 x_2 \\ -\nu_3 x_3 \end{bmatrix}. \tag{3.203}$$

The non-dominant part of the vector field is characterized by a non-dominant exponent $q = 1$. That is, $\mathbf{f}^{(1)}(\boldsymbol{\alpha}^{\mathbf{p}}\mathbf{x}) = \boldsymbol{\alpha}^{\mathbf{p}+q-1}\mathbf{f}^{(1)}(\mathbf{x})$. There are four different balances associated with the exponents $\mathbf{p}$: $\boldsymbol{\alpha}^{(1)} = (1, 1, 1)$, $\boldsymbol{\alpha}^{(2)} = (-1, -1, 1)$, $\boldsymbol{\alpha}^{(3)} = (1, -1, -1)$, and $\boldsymbol{\alpha}^{(4)} = (-1, 1, -1)$. The companion system reads

$$X_1' = X_1 - \omega X_2 X_4 - \nu_1 X_1 X_4 - X_2 X_3, \tag{3.204.a}$$
$$X_2' = X_2 + \omega X_1 X_4 - \nu_2 X_2 X_4 - X_1 X_3, \tag{3.204.b}$$
$$X_3' = X_3 - \nu_3 X_3 X_4 - X_1 X_2, \tag{3.204.c}$$
$$X_4' = X_4. \tag{3.204.d}$$

This system has 5 fixed points, one at the origin and the others given by $\mathbf{X}_*^{(i)} = (\boldsymbol{\alpha}^{(i)}, 0),\ i = 2, \ldots, 5$. The linear eigenvalues at the origin are $\boldsymbol{\lambda} = (-\mathbf{p}, q)$, and the linear eigenvalues at the other fixed points are the Kovalevskaya exponents of the four different balances together with the non-dominant exponent q. Around each fixed point, we compute $J = D\mathbf{F}(\mathbf{X}_*)$ and find $\boldsymbol{\rho} = (-1, 2, 2, 1)$. We can now compute the normal form around each fixed point. For instance, consider $\mathbf{X}_* = \mathbf{X}_*^{(1)} = (1, 1, 1, 0)$. First, we translate the fixed point at the origin of the new system and then we diagonalize the system, by the linear transformation

$$\mathbf{X} - \mathbf{X}_* = M\mathbf{Y}, \tag{3.205}$$

where $M \in \mathrm{GL}(n, \mathbb{R})$ is chosen such that $M^{-1}JM = \mathrm{diag}(\boldsymbol{\rho})$ to obtain

$$J = \begin{bmatrix} 1 & -1 & -1 & -\omega - \nu_1 \\ -1 & 1 & -1 & \omega - \nu_2 \\ -1 & -1 & 1 & -\nu_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.206}$$

and

$$M = \begin{bmatrix} 1 & -1 & -1 & \omega - \frac{1}{2}(\nu_3 + \nu_2 - \nu_1) \\ 1 & 1 & 0 & -\omega - \frac{1}{2}(\nu_3 - \nu_2 + \nu_1) \\ 1 & 0 & 1 & -\frac{1}{2}(\nu_2 - \nu_3 + \nu_1) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.207}$$

The new system for $\mathbf{Y}$ is

$$\mathbf{Y}' = \mathrm{diag}(\boldsymbol{\rho})\mathbf{Y} + \mathbf{G}(\mathbf{Y}), \tag{3.208}$$

where $\mathbf{G}$ is quadratic in $\mathbf{Y}$. We now can look for a normalizing transformation, that is, a near identity transformation $\mathbf{Y} = \mathbf{Z} + \mathbf{P}(\mathbf{Z})$ where $\mathbf{P}$ is a formal power series starting with quadratic terms. We choose $\mathbf{P}(\mathbf{Z})$ so as to eliminate all non-resonant terms in the vector field for $\mathbf{Z}$. Resonant monomials for the jth equation have the form $\mathbf{Z}^{\mathbf{i}}$ where $\boldsymbol{\rho}.\mathbf{i} = \rho_j$ and $|\mathbf{i}| > 1$. The normal form of system (3.208) around $\mathbf{Y} = 0$ reads

$$Z_1' = -Z_1 + O(\mathbf{Z}^3), \tag{3.209.a}$$
$$Z_2' = 2Z_3 + 3C_{21}^{(1)} Z_4^2 + O(\mathbf{Z}^3), \tag{3.209.b}$$
$$Z_3' = 2Z_3 + 3C_{22}^{(1)} Z_4^2 + O(\mathbf{Z}^3), \tag{3.209.c}$$
$$Z_4' = Z_4, \tag{3.209.d}$$

where $C_{21}^{(1)}$ and $C_{21}^{(1)}$ are the compatibility conditions of the Painlevé test given by

$$C_{21}^{(1)} = (\nu_1 - \nu_2)(2\omega + \nu_1 - \nu_2) + \nu_3(\nu_1 + \nu_2 - 2\nu_3) \tag{3.210.a}$$
$$C_{22}^{(1)} = (\nu_2 - \nu_3)^2 + \omega\,(3\,\nu_3 - 2\nu_2 - 4\nu_1) + \nu_1(\nu_2 + \nu_3 - 2\nu_1). \tag{3.210.b}$$

The other balances give the same normal form with different coefficients. Around the fixed point $\mathbf{X}_*^{(2)} = (-1, -1, 1, 0)$, we have $\boldsymbol{\alpha}^{(2)}$ and we find the same coefficient as for the first balance. That is, $C_{21}^{(2)} = C_{21}^{(1)}$ and $C_{22}^{(2)} = C_{22}^{(1)}$. The coefficients for $\boldsymbol{\alpha}^{(3)}$ and $\boldsymbol{\alpha}^{(4)}$ are

$$C_{21}^{(3,4)} = (\nu_1 - \nu_2)(2\omega - \nu_1 + \nu_2) - \nu_3(\nu_1 + \nu_2 - 2\nu_3), \tag{3.211.a}$$
$$C_{22}^{(3,4)} = (\nu_2 - \nu_3)^2 - \omega(3\nu_3 - 2\,\nu_2 - 4\nu_1) + \nu_1(\nu_2 + \nu_3 - 2\nu_1). \tag{3.211.b}$$

According to Proposition 3.7, a necessary condition for the system to have the Painlevé property is that the companion system is linearizable around all its fixed points. That is, we require $C_{2i}^{(j)} = 0$ for $i = 1, 2;\ j = 1, 2, 3, 4$. This leads to the following cases:

1. If $\nu_1 = \nu_2 = \nu_3 = 0$, the system has two first integrals

$$I_1 = x_1^2 + x_2^2 + 2x_3^2, \quad I_2 = x_1^2 - x_2^2 + 4x_3, \tag{3.212}$$

   and the motion takes place on the intersection of the two levels sets $I_i = C_i$ (see Figure 3.3) and the general solution can be written in terms of elliptic functions.

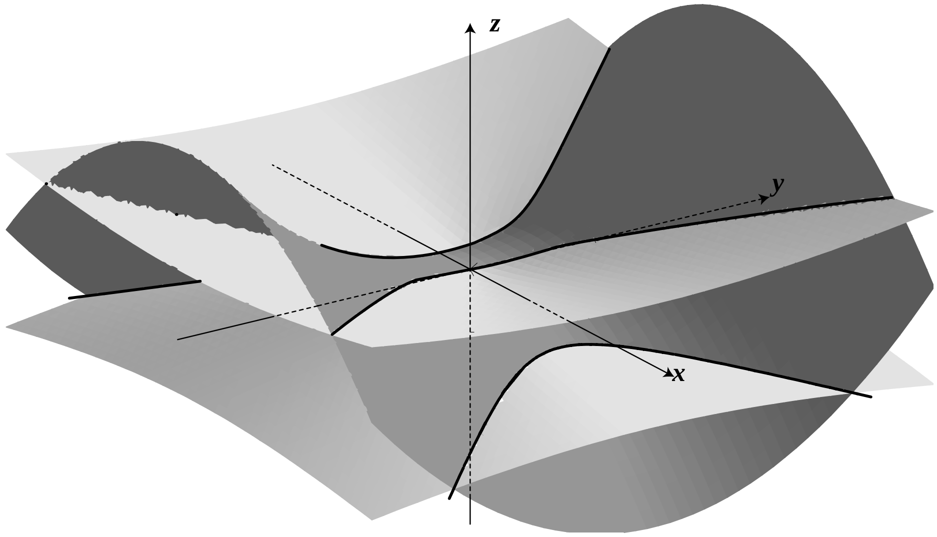

Figure 3.3: The level sets $I_1 = -1$ and $I_2 = 4$ of the first integrals of the Rabinovich system (3.202) when $\nu_1 = \nu_2 = \nu_3 = 0$. The motion takes place on the intersection of a cone with a hyperbolic paraboloid.

2. If $\omega = 0$ and $\nu_i = \nu$, the linear diagonal term can be eliminated by the transformation $y_i = x_i \exp(\nu t)$ and $s = \exp(-\nu t)$. In the new variables y_1, y_2, y_3, the system is equivalent to the previous case.

3. If $\omega = 0$, $\nu_1 = \nu_2$, $\nu_3 = 0$ (and circular permutations), there exists one first integral $I = \exp(2\nu_1 t)(x_1^2 - x_2^2)$ and the reduced system can be integrated in terms of the third Painlevé transcendent.

There exist also partially integrable cases when the system does not have the Painlevé property. If $\omega \neq 0$ and $\nu_i = \nu$, there is a time-dependent first integral $I = \exp(2\nu t)(x_1^2 - x_2^2 - 2x_3^2)$. Note that in this case, one of the compatibility conditions is satisfied. ■

Example 3.21 Resonances between Kovalevskaya exponents. Consider again the fourth order system (3.175) for which Painlevé test #2 was insufficient to reach a conclusion. We first write the equation as a system,

$$\dot{x}_1 = x_2, \tag{3.213.a}$$
$$\dot{x}_2 = x_3, \tag{3.213.b}$$
$$\dot{x}_3 = x_4, \tag{3.213.c}$$
$$\dot{x}_4 = -x_1 x_4 + 2x_3 x_2, \tag{3.213.d}$$

and apply the companion transformation with $\mathbf{p} = (-1, -2, -3, -4)$ to obtain

$$X_1' = X_1 + X_2, \tag{3.214.a}$$
$$X_2' = 2X_2 + X_3, \tag{3.214.b}$$
$$X_3' = 3X_3 + X_4, \tag{3.214.c}$$
$$X_4' = 4X_4 + X_1 X_4 + 2X_3 X_2. \tag{3.214.d}$$

As expected, the linear eigenvalues around $\mathbf{X}_* = (-12, 12, -24, 72)$ are $\boldsymbol{\rho} = (-1, -2, -3, 4)$. The unstable manifold of this fixed point is analytically linearizable since there is only one positive eigenvalue. The stable manifold is characterized by the three negative eigenvalues. The resonance condition $\rho_i = m_1\rho_1 + m_2\rho_2 + m_3\rho_3$ leads to three possibilities: *(i)* $i = 2$ and $m_1 = 2,\ m_1 = m_3 = 0$; *(ii)* $i = 3$ and $m_1 = 1,\ m_2 = 1,\ m_3 = 0$; or *(iii)* $i = 3$ and $m_1 = 3,\ m_2 = m_3 = 0$. Therefore, in order to check that the stable manifold is linearizable the normal form to order 4 must be computed. The normal form is obtained by first translating the fixed point to the origin, second diagonalizing the linear part and third applying a series of near identity transformations. To fourth order, the normal form reads

$$Y_1' = -Y_1 + 36Y_2Y_3Y_4 + \mathrm{O}(\mathbf{Y}^4), \tag{3.215.a}$$
$$Y_2' = -2Y_2^2 - 60Y_3^2Y_4 + \mathrm{O}(\mathbf{Y}^4), \tag{3.215.b}$$
$$Y_1' = -3Y_1 + \mathrm{O}(\mathbf{Y}^4), \tag{3.215.c}$$
$$Y_2' = 4Y_4 + \mathrm{O}(\mathbf{Y}^4). \tag{3.215.d}$$

The dynamics on the unstable manifold is obtained by setting $Y_1 = Y_2 = Y_3 = 0$ and is obviously linear to order 4 (hence to all orders). The dynamics on the stable manifold is obtained, similarly, by setting $Y_4 = 0$, which is again linear, and we conclude that the stable manifold also can be analytically linearized. In terms of the original system, this means that the local series with either ascending or descending powers are Laurent series and the system passes Painlevé test #2. However, the system does not have the Painlevé property since the normal form is not linear. This implies that the general solution of the original system will exhibit movable logarithmic branch points. ■

Example 3.22 A modified Duffing equation. Consider the planar vector field (Conte, 1999)

$$\dot{x}_1 = x_2, \tag{3.216.a}$$
$$\dot{x}_2 = -4x_1x_2 - 2x_1^3. \tag{3.216.b}$$

This system is weight-homogeneous with a unique balance given by $\mathbf{p} = (-1, -2)$ and $\boldsymbol{\alpha} = (1, -1)$. The companion transformation $\mathbf{x} \to \tau^{\mathbf{p}}\mathbf{X}$ and $\tau \to e^s$ leads to the companion system

$$X_1' = X_2 + X_1, \tag{3.217.a}$$
$$X_2' = 2X_2 - 4X_1X_2 - 2X_1^3. \tag{3.217.b}$$

The linear eigenvalues around $\mathbf{X}_* = (1, -1)$ are $\boldsymbol{\rho} = (-1, 0)$. To second order around $\mathbf{X}_*$, the normal form is

$$Y_1' = -Y_1 + \mathrm{O}(\mathbf{Y}^3), \tag{3.218.a}$$
$$Y_2' = -2Y_2^2 + \mathrm{O}(\mathbf{Y}^3). \tag{3.218.b}$$

Therefore, the normal form is not linear which implies that the local series are not pure exponential in s. Written in the original variables, these series are Ψ-series in t with logarithmic terms and we conclude that the original system does not have the Painlevé property. ■

Proposition 3.7 implies that every companion system is formally linearizable around all of its fixed points, except maybe the origin. Moreover, since -1 is always a linear eigenvalue of $\mathbf{X}_*$, all of the fixed points but the origin have resonant eigenvalues. Therefore, the problem of proving the Painlevé property is equivalent to proving that the companion systems can be formally linearized around a resonant fixed point.

We can recast this result in terms of normal forms. Let $\Delta = \mathbf{F}(\mathbf{Y})\partial_{\mathbf{Y}}$, be the vector field of the companion system and $\mathbf{Y} = \mathbf{X} - \mathbf{X}_*$. As explained before, we split this vector field into two parts $\Delta = \Delta_{\text{diag}} + \Delta_{\text{nil}}$ where $[\Delta_{\text{diag}}, \Delta_{\text{nil}}] = 0$ and $\theta^{-1}\Delta_{\text{diag}}\theta = \Delta_{\text{lin}}$. The Painlevé property implies that for each companion system

$$\Delta_{\text{nil}} = 0, \tag{3.219}$$

and $\rho_i, p_i \in \mathbb{Z}\ \forall\ i$.

A digression: The problem of the center.

The problem of proving the Painlevé property of a given analytic vector field has a classical analogue in the theory of dynamical systems, the problem of the center. Consider a planar vector field $\delta = \mathbf{f}(\mathbf{x})\partial_{\mathbf{x}}$ with imaginary eigenvalues $(\pm i)$ at $\mathbf{0}$:

$$\dot{x}_1 = x_2 + f_1(x_1, x_2), \tag{3.220.a}$$
$$\dot{x}_2 = -x_1 + f_2(x_1, x_2). \tag{3.220.b}$$

For a given $\mathbf{f}$, the problem of the center consists of proving the existence of a center at the origin. That is, we have to show that the origin is a fixed point surrounded by an open set of periodic orbits. To do so, we look for a near-identity change of variables that removes all of the nonlinear terms. That is, δ has a center at $\mathbf{0}$ if and only if it can be formally linearized around $\mathbf{0}$. That is,

$$\theta^{-1}\delta\theta = \delta_{\text{lin}}, \tag{3.221}$$

which, equivalently implies that $\delta_{\text{nil}} = 0$. Moreover, there exists a formal first integral provided by θ. The main difficulty in the problem of the center resides in proving the existence of a formal linearizing power series. Indeed, there is no a priori bound on the degree of nonlinear resonant terms. That is, if we can linearize the system up to degree N (that is, if all monomial terms of degree less or equal to N can be removed by polynomial changes of variables), there is no guarantee that the degree $N+1$ also can be linearized. Hence, in general, proving that a fixed point is a center is not a finite decision procedure. Conversely, proving that a fixed point is not a center is a finite decision procedure for it is enough to show that some monomials of degree N cannot be removed by formal power series change of variables. However, the number of steps is not known.

This analogy shows again that the Painlevé property cannot be, in general, fully tested by local analyses since there is no general finite algorithmic decision procedure to check that $\Delta_{\text{nil}} = 0$.

3.10 The weak-Painlevé conjecture

The weak-Painlevé conjecture was introduced to include integrable systems which possess algebraic branch points (Ramani *et al.*, 1982; Grammaticos *et al.*, 1983; Grammaticos *et al.*, 1984; Ramani *et al.*, 1989). In its original form, it only concerns Hamiltonian systems with kinetic diagonal parts of the form

$$H = \frac{1}{2}(p_1^2 + \ldots + p_n^2) + V(x_1, \ldots, x_n). \tag{3.222}$$

Under certain conditions on the potential V, all of the solutions of this Hamiltonian system can be expanded in Puiseux series. Is this information sufficient to conclude that the system is Liouville integrable?

Example 3.23 Algebraic branch points in a nonintegrable Hamiltonian. All the solutions of the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + x_2^4 + \frac{3}{4}x_1^2x_2^2 + \frac{3}{1088}x_1^4, \tag{3.223}$$

can be expanded in Puiseux series (Yoshida *et al.*, 1987b). In particular, there is an expansion of $\mathbf{x} = (x_1,\ x_2)$ given by

$$\mathbf{x} = (t - t_*)^{-1}\sum_{i=0}^{\infty}\mathbf{a}_i(t - t_*)^{i/2}, \tag{3.224}$$

with $\mathbf{a}_8$ and $\mathbf{a}_9$ arbitrary (Kovalevskaya exponents $\rho = 4$, $\rho = 9/2$). However, this system is nonintegrable and exhibits large-scale chaos. ■

The solutions of some Liouville integrable Hamiltonian systems have algebraic branch points.

Example 3.24 Algebraic branch points in an integrable Hamiltonian. As an example, consider the Hamiltonian (Dorizzi *et al.*, 1983)

$$H = \frac{1}{2}(p_1^2 + p_2^2) + x_2^5 + x_1^2x_2^3 + \frac{3}{16}x_1^4x_2. \tag{3.225}$$

This system has two balances defining two Puiseux solutions for $\mathbf{x} = (x_1,\ x_2)$ of the form

$$\mathbf{x} = (t - t_*)^{-2/3}\sum_{i=0}^{\infty}\mathbf{a}_i(t - t_*)^{i/3}. \tag{3.226}$$

For the first Puiseux solution, $\mathbf{a}_5$ and $\mathbf{a}_{10}$ are arbitrary with positive Kovalevskaya exponents $\rho = 5/3$ and $\rho = 10/3$; while for the second one $\mathbf{a}_{12}$ and $\mathbf{a}_{10}$ are arbitrary with Kovalevskaya exponents $\rho = 5/3$, $\rho = 10/3$. ■

What is the difference between system (3.223) and system (3.225)? For system (3.223) the denominator of the Kovalevskaya exponent is 2 while the denominator of the leading exponent is 1. For the second system, both the leading exponents and the Kovalevskaya exponents have a denominator equals to 3. These observations led Ramani, Dorizzi and Grammaticos (1982)to introduce the concept of a *natural denominator*. More precisely the weak-Painlevé property is defined as follows.

Definition 3.5 The Hamiltonian system $H = \frac{1}{2}(p_1^2 + \ldots + p_m^2) + V(\mathbf{x})$ has the *weak-Painlevé property* if the general solution can be locally expanded in Puiseux series whose Kovalevskaya exponents and dominant exponents share the same least common denominator.

Let r be the least common denominator of the dominant exponents. That is, all p_i can be written p'_i/r with $p'_i \in \mathbb{Z}$, $i = 1, \ldots, n$ where $n = 2m$ and the system has the weak-Painlevé property if the solutions can be expanded in Puiseux series and all of the Kovalevskaya exponents for all the dominant balances can be written $\rho_i = \rho'_i/r,\ i = 2, \ldots, n$ where $\rho'_i \in \mathbb{Z}$. In particular, if $r = 1$, the system satisfies the usual Painlevé test (#2).

The weak-Painlevé conjecture: *If the Hamiltonian* $H = \frac{1}{2}(p_1^2 + \ldots + p_m^2) + V(x_1, \ldots, x_m)$ *has the weak-Painlevé property, then it is Liouville integrable.*

The converse statement is not true even if we only restrict it to Liouville integrable systems with quasimonomial first integrals as shown in the next example.

Example 3.25 The Holt Hamiltonian. Consider, for instance, the Holt Hamiltonian (Holt, 1982)

$$H = \frac{1}{2}(p_1^2 + p_2^2) + x_2^2 x_1^{-2/3} + \frac{3}{4}\lambda x_1^{4/3} x_2. \tag{3.227}$$

This system is Liouville integrable for $\lambda \in \{1, 6, 16\}$. However, it supports expansions whose dominant balances are $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$ with $\mathbf{p} = (-6,-6,-7,-7)$, $\boldsymbol{\alpha} = (\pm 65856\sqrt{3}/\lambda^3, -98784/\lambda^3, -6\alpha_1, -6\alpha_2)$, and with Kovalevskaya exponents $\mathcal{R} = \{-1, 14, 13/2 \pm 1/2\, i\sqrt{167}\}$. The presence of irrational Kovalevskaya exponents imply that this system does not have the Painlevé property. ■

It will be shown in Section 4.16.2 that a change of variables can be introduced in order to transform the Puiseux series into a Laurent series. However, this change of variables cannot be performed for all systems with the weak-Painlevé property. When such a transformation exists, it provides a direct link between systems with Puiseux series and those with the Painlevé property (Goriely, 1992).

3.11 Patterns of singularities for nonintegrable systems

A general system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, will not pass the Painlevé test. However, under some assumptions on the leading part of the vector field at the singularity, some valuable information from the Ψ-*series expansion* (the local series with logarithmic terms) on the singularity structure may still be obtained.

The relationship between nonintegrability and the singularity patterns in complex time has been explored in hydrodynamics (Frisch & Morf, 1981) where it was shown that intermittency is related to the distribution of the singularities in complex time near the real axis. Other nonintegrability phenomena related to singularity structure have been observed for physical problems such as for flame fronts (Thual *et al.*, 1985) or for the Burgers equation (Bessis & Fournier, 1984). Also, in a series of papers Bountis and co-workers showed an intriguing pattern of singularities in the form of *chimneys* for Hamiltonian system with algebraic singularities (Bountis, 1992; Bountis *et al.*, 1991). However, to date there is no comprehensive understanding of the relation between nonintegrable dynamics and singularity patterns in the complex plane. Nevertheless, there are two main results on singularity clustering depending on the irrationality of the Kovalevskaya exponents and the logarithmic terms in the local series solutions.

3.11.1 Kovalevskaya fractals

The irrationality of the Kovalevskaya exponents can be used to show that complex time singularities tend to cluster on self-similar curves. This beautiful construction was first proposed by by Chang, Tabor, Weiss and Greene (1981; 1982; 1983) and further investigated by Yoshida (1984). The construction is as follows. Assume that a weight-homogeneous system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, has a scale-invariant solution $\mathbf{x} = \boldsymbol{\alpha}t^{\mathbf{p}}$ and a set of Kovalevskaya exponents with at least one non-real exponent, say $\rho_2 = \rho = \lambda + i\mu$, $\lambda > 0$. We set the arbitrary constants associated with the other Kovalevskaya exponents to zero in the general local solution around the singularity corresponding to the balance

$\{\boldsymbol{\alpha}, \mathbf{p}\}$ to obtain

$$\mathbf{x} = \boldsymbol{\alpha}(t-t_*)^{\mathbf{p}}\mathbf{P}(\boldsymbol{\gamma}(t-t_*)^{\rho}). \tag{3.228}$$

According to Theorem 3.3, the series $\mathbf{P}(\mathbf{y})$ have a non-zero radius of convergence. Therefore, for a fixed arbitrary constant, there exists, on the convergence circle, a value of $\mathbf{y} = \mathbf{y}_*$ where at least one of the components of $\mathbf{x}$ is singular (say the i-th component). Let $t_0 \in \mathbb{C}$ be the time at which this singularity appears $\boldsymbol{\gamma}(t_0 - t_*)^{\rho} = \mathbf{y}_*$. In turn, there exists a sequence of values t_n where the series diverges defined by

$$(t_n - t_*) = e^{\frac{2\pi i n}{\rho}}(t_0 - t_*), \quad n = 1, 2, \dots \tag{3.229}$$

Therefore, if we start with a singularity t_* there exist other singularities $t_0, t_1, t_2, \dots$. These singularities are located on a logarithmic spiral in such a way that the triangles of vertices (t_n, t_{n+1}, t_{n+2}) are self-similar. Given t_n and t_{n+1}, the next singularity is placed at an angle θ and a distance r^n such that $re^{i\theta} = \exp(\frac{2\pi i}{\rho})$, that is, $r = \exp(\frac{2\pi\mu}{|\rho|^2})$ and $\theta = \frac{2\pi\lambda}{|\rho|^2}$. An example of such a construction is given in Figure 3.4. These self-similar structures have a fractal structure and have been called *Kovalevskaya*

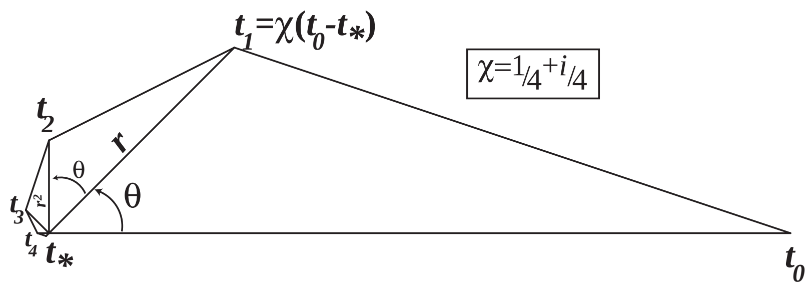

Figure 3.4: The construction of the self-similar Kovalevskaya fractals ($\chi = re^{i\theta}$).

fractals. The Hénon-Heiles system and the Kuramoto model for the evolution of flame fronts are examples of nonintegrable systems where such patterns of singularities have been identified both numerically and theoretically (Thual *et al.*, 1985). These structures were thought to form a natural boundary since each of the singularity $t_1, t_2, \dots$ also have a self-similar spiral of singularities around them. However, it was found that these different singularities may not lie on the same Riemann sheet, and that the spiral formed is only the projection of isolated singularities on the complex plane (Bessis, 1990).

3.11.2 Singularity clustering.

A different type of clustering may be obtained from the logarithmic terms. For simplicity, we assume that for a given balance, the Kovalevskaya exponents, $\boldsymbol{\rho}$, and dominant exponents, $\mathbf{p}$, are integers. Moreover, all of the compatibility conditions are satisfied except for one (say at $\rho = r$). In this case, we know that the solutions can be formally expanded in the series (Bender & Orszag, 1978; Tabor & Weiss, 1981)

$$\mathbf{x} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbf{a}_{ij}(t-t_*)^{i-p}\left((t-t_*)^r \log(t-t_*)\right)^j. \tag{3.230}$$

The idea here is to introduce an infinite set of generating functions, Ψ_i, and a new variable, $\xi = (t-t_*)^r \log(t-t_*)$, to rewrite these bi-infinite series as

$$\mathbf{x} = \sum_{i=0}^{\infty} \Psi_i(\xi)(t-t_*)^{i-p}, \tag{3.231.a}$$

$$\Psi_i(\xi) = \sum_{j=0}^{\infty} a_{ij}\xi^j. \tag{3.231.b}$$

Now, it is possible to study the asymptotic behavior of these series for $|\xi| \gg |(t-t_*)|$. In this limit, an ODE for $\Psi_0(\xi)$ is obtained, the solution of which gives a local picture of the clustering of the singularities (Fournier *et al.*, 1988; Levine & Tabor, 1988; Bountis *et al.*, 1993). This procedure can be performed only if the weight-homogeneous part of the vector field is itself integrable. That is, the equation for Ψ_0 can be effectively solved if and only if the leading part of the vector field is integrable.

3.12 Finite time blow-up

The problem of finding finite time blow-up for partial differential equations is a most active domain of research and literally hundreds of paper have been written on the subject (Palais, 1988; Glangetas & Merle, 1994; Doering & Gibbon, 1995; Ohta, 1995; Peral & Vazquez, 1995). In this section we consider the more modest analog problem for ODEs (Stuart & Floater, 1990; Matsumo, 1992). The existence of finite time singularities is relevant in many areas of applied mathematics: in dynamical systems theory, the absence of finite time blow-up is important in order to prove the boundedness of solutions (Coomes, 1989); in Hamiltonian dynamics, Liouville integrability requires that the flows associated with the first integrals are defined for all time, that is, they do not exhibit finite time blow-up (Flaschka, 1988; Marsden & Ratiu, 1994); in fluid dynamics, many simplified models reduce to dynamical systems for which the spontaneous formation of singularities can be tested (Hocking *et al.*, 1972; Frisch & Morf, 1981; Vieillefosse, 1982; Ohkitani, 1993); in incompressible magneto-hydrodynamics, the formation of singularities has been shown to have important physical implications such as the occurrence of solar flares in the solar dynamo problem (Klapper *et al.*, 1996); and in the theory of thin-film and filaments singularities are associated with pinch-off and rupture (Zhang & Lister, 1999; Bernoff *et al.*, 1998; Witelski & Bernoff, 1999).

Consider a real analytic system of ODEs:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{3.232}$$

Following Section 3.2, the general solution of (3.232) is a solution that contains n arbitrary constants. The *particular solutions* are obtained from the general solution by setting some of the arbitrary constants to a given value. The particular solutions describe the evolution of restricted subsets of initial conditions. Let $\mathbf{x} = \mathbf{x}(t; c_1, \dots, c_k)$ denote a solution depending on k arbitrary constants. This solution is a general solution when $k = n$. We also denote $\mathbf{x} = \mathbf{x}(t; \mathbf{x}_0)$ the solution based on the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

The system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ exhibits a *finite time blow-up* if there exist both $t_* \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$ such that for all $M \in \mathbb{R}$, there exists an $\varepsilon > 0$ such that for $|t-t_*| < \varepsilon$, we have

$$\| \mathbf{x}(t; \mathbf{x}_0) \| > M, \tag{3.233}$$

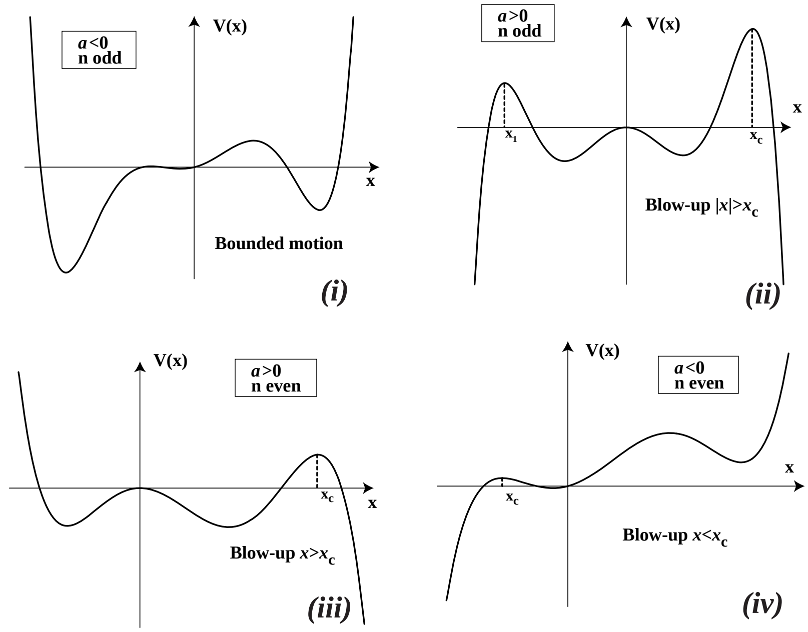

Figure 3.5: The different possible potential configurations for Equation (3.235).

where $\|\ .\ \|$ is any l^p norm. The blow-up is *forward* in time if $t_* > t_0$ and *backward* if $t_* < t_0$. Equivalently, we use

$$\lim_{t \to t_*} \| \mathbf{x}(t, \mathbf{x}_0) \| \to \infty, \tag{3.234}$$

to denote such a blow-up. There are many interesting questions related to the existence of finite time blow-ups in ODEs. Among others: *(i)* Are there sets of initial conditions $S_0^{(k)}$ of dimension k such that $\forall\ \mathbf{x}_0 \in S_0^{(k)}$, there exists a real t_* such that $\| \mathbf{x}(t, \mathbf{x}_0) \| \to \infty$ as $t \to t_*$ (with $t < t_*$)? *(ii)* Are there open sets of initial conditions with the same property as that outlined in *(i)*? *(iii)* What is the nature of those sets? *(iv)* Where does blow-up occur in phase space? Some of these questions were investigated by Goriely and Hyde (1998; 2000) in a more restrictive setting.

Example 3.26 Blow-up in a one-degree-of-freedom system. To illustrate the problem, we consider a one-degree-of-freedom Hamiltonian system with polynomial potential

$$\ddot{x} = ax^n + g(x), \tag{3.235}$$

where $g(x)$ is a polynomial of degree less than n with $g(0) = 0$, $n \geq 3$ and $a \neq 0$. This system has a Hamiltonian $H = \dfrac{\dot{x}^2}{2} + V(x)$ with potential $V(x) = -a\dfrac{x^{n+1}}{n+1} - \displaystyle\int^x g(x)dx$. Depending on the parity of n and the sign of a, this system can exhibit finite time blow-up. Already, it can be seen that not all trajectories diverge to infinity. For instance, the fixed point $\mathbf{x} = \mathbf{0}$ is a particular solution which does not exhibit a finite time blow-up. This is why we are interested in proving the existence of an open set of initial conditions rather than proving that all initial conditions lead to a blow-up. The analysis of the singularities of this system is straightforward when we consider the graph of the potential function. Depending on the parity of n and the sign of a, four different cases can be discussed (see Figure 3.5):

(i) if n is odd and a is negative (Figure3.5(i)), then all orbits are bounded in phase space and there is no possibility for a blow-up; *(ii)* if n is odd and a is positive, then by choosing $|x|$ large enough ($|x| > x_c$ in Figure3.5(ii)), an open set of initial conditions $\{x_0, \dot{x}_0\}$ leading to finite time blow-ups can be found easily. Moreover, for these initial conditions, the blow-up time t_* can be explicitly computed as

$$t_* = \int_{x_0}^{\infty} \frac{dx}{\sqrt{2[E - V(x)]}}, \tag{3.236}$$

with $E = H(x_0, \dot{x}_0)$. Finally, *(iii)* and *(iv)*, if the potential is odd (n even), then independently of the choice of a, there always exists a critical value x_c that leads to a blow-up for $x > x_c$ (for $a > 0$) or $x < x_c$ (for $a < 0$). In this case however, the blow-up occurs only in one quadrant of the phase space. The blow-up time can be obtained by considering (3.236). Finally, we observe that the lower-order terms, denoted by $g(x)$, do not change the main result; blow-up can be delayed but it cannot be avoided in the entire phase space. One of the possible effects of including the lower terms $g(x)$ in the system is to create regions in phase space where the solution is bounded; for instance, the choice $\dot{x}_0 = 0$ and $|x_0| < x_1$ for the potential in Figure 3.5(ii) leads to periodic orbits or fixed points.

Now, we can compare this analysis with the one performed locally around the singularities. The asymptotic expansions around a singularity $t_* \in \mathbb{C}$ for the solutions of Equation (3.235) is

$$x = \alpha(t - t_*)^p \left(1 + h((t - t_*)^s)\right), \tag{3.237}$$

where $h(t - t_*)$ is, in general, a Taylor series in its argument and s is a rational number. Its explicit form can be found but is not relevant here. The leading term $\alpha(t - t_*)^p$ is related to a and n in the following way:

$$p = \frac{2}{1-n}, \qquad \alpha^{n-1} = \frac{-2(1+n)}{a(1-n)^2}. \tag{3.238}$$

The asymptotic form of the solution around the singularities depends only on the dominant term ax^n and not on the lower order terms. Depending on the sign of a and the parity of n, the leading coefficient α can be real or complex. Let $\beta = (-1)^p \alpha$, if n is even, there always exists a root[12] $\beta = \left(\dfrac{2(1+n)}{a(1-n)^2}\right)^{\frac{1}{n-1}} \in \mathbb{R}$. If a is positive and n is odd, there are two such real roots $\beta = \pm\left(\dfrac{2(1+n)}{a(1-n)^2}\right)^{\frac{1}{n-1}} \in \mathbb{R}$. However, if a is negative and n odd, there is no real root for β. These observations indicate that *finite time blow-up occurs whenever one of the leading coefficients of the asymptotic series is real.* Moreover, when blow-up occurs in two different quadrants of phase space (Figure3.5(*ii*)), two different series with real leading coefficients can be found. ■

This simple example seems to indicate that there is a simple connection between the real-valuedness of the leading coefficient and the occurrence of blow-up. Here again, we use the notion of companion systems previously developed to build a set of initial conditions on the unstable manifold of a real fixed point for the companion system and show that, under certain conditions, this set is mapped to a real set of initial conditions blowing up in the original phase space (Goriely, 2001). We note that,

[12] The root $a = c^{1/b}$ for $c > 0$ and $b \in \mathbb{R}$, is the positive real number a such that $a^b = c$.

whenever $\boldsymbol{\alpha} \in \mathbb{R}^n$, the corresponding local series solution (see Equation (3.87))

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{\mathbf{i},\ |\mathbf{i}|>1}^{\infty} \mathbf{c_i} \tau^{\boldsymbol{\rho}.\mathbf{i}} \right), \qquad \boldsymbol{\rho}.\mathbf{i} = \sum_{j=1}^{n+1} \rho_j i_j, \tag{3.239}$$

around the singularities t_* is real (for real t_* and real arbitrary constants).

Theorem 3.8 *Consider a real analytic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and assume that it has a balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$ with $(k-1)$ Kovalevskaya exponents with positive real parts (excluding $\rho_{n+1} = q$) and let $\boldsymbol{\beta} = (-1)^{\mathbf{p}}\boldsymbol{\alpha}$. Then, if $\boldsymbol{\alpha} \in \mathbb{R}^n$ (resp. $\boldsymbol{\beta} \in \mathbb{R}^n$), there exists a k-dimensional manifold S_0^k of initial conditions leading to a finite backward (resp. forward) time blow-up.*

Proof. We consider the case of a backward finite time blow-up and define $\tau = (t - t_*)$. The case of a forward time blow-up is obtained similarly by defining $\tau = (t_* - t)$ Consider the companion system associated with the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$ where $\boldsymbol{\alpha} \in \mathbb{R}^n$. Then, $\mathbf{X}_* = (\boldsymbol{\alpha}, 0)$ is a real fixed point of a real analytic system. Let $\mathbf{X}_0 \in W_{\mathrm{u}}(\mathbf{X}_*)$ be a point on the unstable manifold of $\mathbf{X}_*$. This point is mapped by the inverse of the companion transformation to a point in the original phase space that blows up in finite time. That is, when $\mathbf{X}(s, \mathbf{X}_0) \rightarrow_{s \rightarrow -\infty} \mathbf{X}_*$, we have

$$\| \mathbf{x}(t, \mathbf{x}_0) \| \underset{t \rightarrow t_*}{\rightarrow} \infty, \tag{3.240}$$

where $\mathbf{x}_0 = \tau_0^{\mathbf{p}} \hat{\mathbf{X}}_0 \in \mathbb{R}^n$ and $t_* = t_0 - X_{0,N} \in \mathbb{R}$, $\tau_0 = t_0 - t_*$. Now, there exists, by the unstable manifold theorem, a k-dimensional manifold of points $\mathbf{X}_0$ such that $\mathbf{X}(s, \mathbf{X}_0) \rightarrow \mathbf{X}_*$ as $s \rightarrow -\infty$. This manifold is mapped to a k-dimensional manifold of blow-up points $\mathbf{x}_0$ in the original phase space. $\square$

In the particular case when the balance is principal, and $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ is real, there exist $(n-1)$ Kovalevskaya exponents with strictly positive real parts and open sets of initial conditions leading to a finite time blow-up. When some of the Kovalevskaya exponents vanish, that is, when the fixed point of the companion system is not hyperbolic, the situation is more difficult to describe. This is again due to the fact that the stability of non-hyperbolic fixed points cannot be fully determined by a linear analysis. However, there is a simple case where blow-ups occur only on some components. If for a given balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$, l components of $\boldsymbol{\alpha}$ are strictly equal to zero and $(k-1)$ Kovalevskaya exponents have positive real parts, there exists a manifold S_0^m of dimension $m \geq k + l$ leading to finite time blow-up. Moreover, the location of the blow-up set in phase space can be obtained from the leading order behavior.

Proposition 3.8 *Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. let S_0^k be the blow-up manifold obtained in Theorem 3.8 and $\mathcal{O}_{\mathrm{sign}(\boldsymbol{\alpha})}$ (resp. $\mathcal{O}_{\mathrm{sign}(\boldsymbol{\beta})}$) be the orthant in phase space determined by the sign of the components of $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$).*[13] *Then, the forward (resp. backward) blow-up occurs in the orthant of $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$), that is, $\mathcal{O}_{\mathrm{sign}(\boldsymbol{\alpha})} \cap S_0^k \neq \emptyset$ (resp. $\mathcal{O}_{\mathrm{sign}(\boldsymbol{\beta})} \cap S_0^k \neq \emptyset$).*

We now discuss the existence of a finite time blow-up in the presence of first integrals. In some instances, first integrals can be used to directly prove the absence of a finite time blow-up. For instance, if a two-dimensional system has a first integral $I = x_1^2 + x_2^2$, there is no possibility of a finite time blow-up ($I = x_{01}^2 + x_{02}^2 = x_1^2 + x_2^2 \in \mathbb{R}$ implies $x_1, x_2 \in \mathbb{R}\ \forall\ t$). If, however, $I = x_1^2 - x_2^2$, then blow-up cannot be ruled out since the solutions may go to infinity in such a way so that the difference

[13] The *orthant* $\mathcal{O}_{\boldsymbol{\epsilon}}$ is the set $\{\mathbf{x} \in \mathbb{R}^n | \epsilon_i x_i > 0, i = 1, \ldots, n\}$.

of the squares remains constant. It is therefore straightforward to obtain the following well-known result.

Proposition 3.9 *Let $I = I(\mathbf{x})$ be a first integral for the real analytic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. If the level sets of I are compact, then there is no finite time-blow-up (backward or forward) for any open set of initial conditions.*

How is this result related to Theorem 3.8? If $I = I(\mathbf{x}, t)$ is a first integral for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then there exists a first integral $I^{(0)} = I^{(0)}(\mathbf{x})$ for the system $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$, where $\mathbf{f}^{(0)}$ is the dominant part of the vector field with respect to the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$. Since the first integral $I^{(0)}$ is constant on all solutions, it is constant on the particular solution $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$. Therefore, $I^{(0)}(\boldsymbol{\alpha}\tau^{\mathbf{p}}) = I^{(0)}(\boldsymbol{\alpha})\tau^d = 0$ implies $I^{(0)}(\boldsymbol{\alpha}) = 0$. However, if the level sets of $I(\mathbf{x})$ are compact, so are the level sets of $I^{(0)}(\mathbf{x})$ and the relation $I^{(0)}(\boldsymbol{\alpha}) = 0$ cannot be satisfied if $\boldsymbol{\alpha} \in \mathbb{R}^n$. So, the fact that $I^{(0)}$ is of definite sign implies that the corresponding balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$ is such that $\mathrm{Im}(\boldsymbol{\alpha}) \neq 0$. This argument provides a proof of a generalization of the previous proposition.

Proposition 3.10 *Let $I^{(0)} = I^{(0)}(\mathbf{x})$ be a first integral of a dominant part of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$. If the level sets of $I^{(0)}$ are compact, then there is no finite time blow-up for an open set of initial conditions.*

Example 3.27 The Rikitake system. The model under consideration was first derived by Rikitake in 1958 and consists of two identical single Faraday-disk dynamos of the Bullard type coupled. It is used by geophysicists as a conceptual model to study the time series of geomagnetic polarity reversals over geological time (Cook & Roberts, 1970; Steeb, 1982; Ershov *et al.*, 1989). It reads,

$$\dot{x} = -\mu x + zy, \tag{3.241.a}$$
$$\dot{y} = -\mu y + (z - a)x, \tag{3.241.b}$$
$$\dot{z} = 1 - bxy. \tag{3.241.c}$$

The traditional Rikitake model is recovered by setting $b = 1$. We now show that depending on the value of b, the solutions of the Rikitake model exhibit (for an open set of initial conditions) finite time blow-up. To do so, we isolate two possible balances. The first one corresponds to the truncation of the vector field: $\mathbf{f}^{(0)} = (zy, zx, -bxy)$ with balance

$$\mathbf{p}^{(1)} = (-1, -1, -1),\ \boldsymbol{\alpha}^{(1)} = (\frac{\pm i}{\sqrt{b}}, \frac{\pm i}{\sqrt{b}}, 1), \quad \text{or} \quad \boldsymbol{\alpha}^{(1)} = (\frac{\pm i}{\sqrt{b}}, \frac{\mp i}{\sqrt{b}}, -1), \tag{3.242}$$

and Kovalevskaya exponents $\boldsymbol{\rho} = (-1, 2, 2)$. The second balance corresponds to the truncation $\mathbf{f}^{(0)} = (zy, -ax, -bxy)$ with balances

$$\mathbf{p}^{(2)} = (-2, -1, -2),\ \boldsymbol{\alpha}^{(2)} = (\frac{\pm 2i}{a\sqrt{b}}, \frac{\mp 2i}{\sqrt{b}}, \frac{-2}{a}), \tag{3.243}$$

and Kovalevskaya exponents $\boldsymbol{\rho} = (-1, 2, 4)$. Therefore, both balances correspond to a general solution and we conclude that if $b < 0$ there exist open sets of initial conditions leading to finite time blow-up. Furthermore, if $b > 0$, there is no finite time blow-up in the Rikitake model (for open set of initial conditions) since no balance $\boldsymbol{\alpha}^{(1)}$, and $\boldsymbol{\alpha}^{(2)}$ are real (Goriely & Hyde, 1998).

Note that even when finite time blow-up is ruled out (for $b > 0$) the solutions are not bounded and can still blow-up in infinite time. For instance, the simple particular solution $x = y = 0, z = t$ is unbounded as $t \to \infty$. ■

Example 3.28 The Vieillefosse model. In order to model the interaction between vorticity and shear in turbulent flow Vieillefosse (1982), introduced a five-dimensional ODE system for which the existence of blow-up implies that the flow of an incompressible and inviscid fluid diverges in a finite time. The system reads

$$\dot{x_1} = -x_3 - x_4, \tag{3.244.a}$$
$$\dot{x_2} = x_4, \tag{3.244.b}$$
$$\dot{x_3} = -\frac{3}{2}x_5 + \frac{1}{2}x_1x_2 - \frac{1}{4}x_1^2, \tag{3.244.c}$$
$$\dot{x_4} = \frac{1}{2}x_5 + \frac{1}{6}x_1x_2 - \frac{1}{3}x_2^2, \tag{3.244.d}$$
$$\dot{x_5} = \frac{1}{3}x_4x_1 - \frac{2}{3}x_2x_4. \tag{3.244.e}$$

This system admits a Ψ-series solution with balance

$$\mathbf{p} = (-2,-2,-3,-3,-4), \quad \boldsymbol{\alpha} = (144, 72, -432, 144, 864), \tag{3.245}$$

and Kovalevskaya exponents $\boldsymbol{\rho} = (-1, 2, 3, 4, 6)$. Since the leading order coefficients are real and there are four positive Kovalevskaya exponents, we conclude that the general solution of this system will exhibit finite time blow-up for some open set of real initial conditions. Moreover, the blow-up occurs on the orthant $\mathcal{O}_{(+,+,-,+,+)}$. ■

Example 3.29 The generalized Toda lattice. We consider finite non-periodic Toda lattices with indefinite metric (Tomei, 1984; Kodama & Ye, 1996). These systems are variations of the classical Toda lattice. Written in Flaschka's variables (Flaschka, 1974), the N-particle system reads

$$\dot{a}_i = s_{i+1}b_i^2 - s_{i-1}b_{i-1}^2, \qquad i = 1, \dots, N, \tag{3.246.a}$$
$$\dot{b}_i = \frac{1}{2}b_i(s_{i+1}a_{i+1} - s_ia_i), \quad i = 1 \dots, N-1, \tag{3.246.b}$$

where $b_0 = b_N = a_{N+1} = 0$ and s_i is either $+1$ or -1. In the particular case where all s_i are $+1$, system (3.246) is the classical Toda lattice. To eliminate the explicit dependence on the signs s_i, we introduce the variables

$$x_i = s_ia_i, \qquad i = 1, \dots, N, \tag{3.247.a}$$
$$y_i = s_is_{i+1}b_i^2, \quad i = 1, \dots, N-1. \tag{3.247.b}$$

The new system reads

$$\dot{x}_i = y_i - y_{i-1}, \qquad i = 1, \dots, N, \tag{3.248.a}$$
$$\dot{y}_i = y_i(x_{i+1} - x_i), \quad i = 1 \dots, N-1. \tag{3.248.b}$$

with $y_0 = y_N = x_{N+1} = 0$. Kodama and Ye (1998) used the complete integrability properties of this system to investigate the occurrence of finite time blow-up. To study the existence of blow-up and the dimension of the blow-up manifolds, we study all possible balances of the form

$$x_i = \alpha_i\tau^{p_i}, \qquad i = 1, \dots, N \tag{3.249.a}$$
$$y_i = \beta_i\tau^{q_i}, \qquad i = 1, \dots, N-1. \tag{3.249.b}$$

Since the system is weight-homogeneous, we have $p_i = -1$ and $q_i = -2$ for all i. One can order the balances by the number of vanishing components of the vector $\boldsymbol{\beta}$. A combinatorial computation shows that the number of balances with k vanishing components of $\boldsymbol{\beta}$ is $\binom{N}{k}$. Hence, the total number of possible balances is equal to $2^{N-1} - 1$. The case where all components of $\boldsymbol{\beta}$ vanish, the first non-vanishing term corresponds to a Taylor series and not a singular solution. The occurrence of a blow-up on open sets of initial conditions can be readily computed for this problem

Proposition 3.11 *Provided that at least one but not all $s_i = -1$, there exists an open set of initial conditions in $\mathbb{R}^{2N-1}$ leading to a finite time blow-up for system (3.246). In this type of blow-up, only three of the $2N-1$ variables blow-up, namely two components of $\mathbf{a}$ and one component of $\mathbf{b}$,.*

Proof. To obtain blow-up for open sets of initial conditions, only the principal balances have to be considered. That is, the balances for which all Kovalevskaya exponents have a positive or null real part. Note that since the exponents $\mathbf{p}, \mathbf{q}$ are integers, the existence of a backward blow-up implies the existence of a forward blow-up. Hence, we only consider backward blow-up. For system (3.247), there exist $(N-1)$ principal balances of the form (3.249) obtained in the following way. Let j be an integer between 1 and $(N-1)$ and choose

$$\beta_i = -\alpha_i = \alpha_{i+1} = -1 \quad \text{if} \quad i = j, \tag{3.250.a}$$

$$\alpha_i = \beta_i = 0 \quad \text{otherwise}, \tag{3.250.b}$$

All of these balances are principal with Kovalevskaya exponents

$$\boldsymbol{\rho} = \{-1, 1(n_1 \text{ times}), 2(n_2 \text{ times}), 3(n_3 \text{ times})\}, \tag{3.251}$$

where $n_1 = N$; $n_2 = N-2, n_3 = 1$ if $j = 1$ or $j = N-1$ and $n_1 = N$, $n_2 = N-3$, $n_3 = 2$ otherwise. The corresponding unstable manifold of the companion system can be used to build open sets of initial conditions for the original variables. Only three of the $2N-1$ variables actually blow-up (namely, x_j, x_{j+1} and y_j, for any given $j < N$), the other variables are analytic functions of τ around the singularity. Now, in terms of the original variables $(\mathbf{a}, \mathbf{b})$, for the balances under consideration, we have $\beta_j = -1$ and $\beta_i = 0$ $i \neq j$. Hence, for any given vector $\mathbf{s} = (s_1, \ldots, s_N)$ with at least one but not all components equal to -1, there exists an entry s_j such that $s_j s_{j+1} = -1$. For this choice of j, the corresponding balance provides a real set of initial conditions in the original phase space. ■

The other balances correspond to a situation where more than three variables blow-up but the blow-up occurs on smaller dimensional manifolds. In the case where all variables blow-up at the same time, we have the following result.

Proposition 3.12 *The Toda system (3.246) exhibits blow-up in all the variables (backward and forward) if and only if N is even and the signs are alternating, that is, $s_i s_{i+1} = -1$, $i = 1, \ldots, N-1$. Moreover, the blow-up manifold is of dimension N.*

Proof. Blow-up occurs in all the variables if and only if there exists a balance (3.249) such that $\alpha_i \neq 0$ for all i. The computation of such a balance shows that $\alpha_i = N + 1 - 2i$ and $\beta_i = i(i-N)$. However, for N odd, $\alpha_{\frac{N+1}{2}} = 0$, which is in contradiction with the assumption that $\alpha_i \neq 0$. Hence, for N odd, the variable $a_{\frac{N+1}{2}}$ does not blow-up in finite time. Since all β_i are strictly negative, we have to choose $s_i s_{i+1} = -1$ to ensure that b_i is real when y_i blows up. If N is even, the $2N-1$ Kovalevskaya exponents are $\boldsymbol{\rho} = \{-N+1, \ldots, -1, 1, \ldots, N\}$, that is, there are N positive Kovalevskaya exponents and the corresponding blow-up manifold is of dimension N. ■ □

3.13 Exercises

3.1 Show that the solution of the equation

$$(\dddot{x} - 2\dot{x}\ddot{x})^2 + 4\ddot{x}^2(\ddot{x} - \dot{x}^2 - 1) = 0, \tag{3.252}$$

admits a single-valued general solution of the form

$$x = a_1 e^{a_2 t} + a_3 t + a_4, \tag{3.253}$$

and find the constants a_i in terms of the arbitrary constants c_1, c_2, c_3. Moreover show that the equation admits a singular solution with a logarithmic branch point. Does this equation have the Painlevé property?

3.2 Show that

$$(1 - t^2)\dot{x}^2 = 1 - x^2, \tag{3.254}$$

represents a family of conics touching the four sides of a square (Ince, 1956, p. 92).

3.3 **Liouville's theorem.** Let $X(t)$ be a fundamental solution of

$$\dot{X} = M(t)X, \tag{3.255}$$

where $M(t)$ is holomorphic in a domain $D \subset \mathbb{C}$ around $t = t_0$. Show that the determinant of the fundamental solution $\chi(t) = \det X(t)$ is given by

$$\chi(t) = \chi(t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr}(M(s))ds\right). \tag{3.256}$$

Moreover, show that $\chi(t)$ is holomorphic in D and that the formula is still valid if M is singular at t_0. (Hint: expand $X(t + \Delta t)$ to first order in Δt and compute the determinant of both sides and take the limit $\Delta t \to 0$, use the identity $\det(\mathrm{I} + hM) = 1 + h\operatorname{tr}(M) + O(h^2)$ to obtain a differential equation for the determinant) (Hille, 1976, p. 324).

3.4 Consider the system $\dot{\mathbf{x}} = \frac{1}{t}R\mathbf{x},\ \mathbf{x} \in \mathbb{C}^n$. The general solution is given by $\mathbf{x} = t^R\mathbf{C}$ (see Proposition 3.2). *(i)* Show that if H is semi-simple, this general solution is equivalent to

$$\mathbf{x} = \sum_{i=1}^{n} c_i \boldsymbol{\beta}^{(i)} t^{\rho_i}, \tag{3.257}$$

where $\boldsymbol{\beta}^{(i)}$ is the eigenvector of H of eigenvalue ρ_i ($\boldsymbol{\beta}^{(i)}$ and $\boldsymbol{\beta}^{(j)}$ are chosen to be linearly independent if $\rho_i = \rho_j$) and c_i are arbitrary constants. *(ii)* If R is not semi-simple, then $R = S+N$ where S is a semi-simple matrix with the same eigenvalues as R and N is a nilpotent matrix of order $k \le n$ which commutes with S. (A matrix N is *nilpotent* of order k if $N^{k-1} \neq 0$ and $N^k = 0$) Use this decomposition to show that the general solution can be written

$$\mathbf{x} = P\operatorname{diag}(t^{\boldsymbol{\rho}})P^{-1}\left(\sum_{i=0}^{k} \frac{N^k \log t}{(k-1)!}\right)\mathbf{C}, \tag{3.258}$$

where $P \in \mathrm{GL}(n, \mathbb{C})$ is such that $P^{-1}SP = \operatorname{diag}(\boldsymbol{\rho})$ and $\operatorname{diag}(t^{\boldsymbol{\rho}}) = \operatorname{diag}(t^{\rho_1}, \ldots, t^{\rho_n})$.

3.5 Perform the singularity analysis for the equation

$$\ddot{x} = P_d(x), \tag{3.259}$$

where P_d is a polynomial of degree $d > 1$. Show that both the dominant exponent and the second Kovalevskaya exponent ρ is determined by d. Show that the non-dominant exponents $q^{(i)}$ are determined by the non-vanishing terms of degree less than d. What is the first integral of the system? How is the weighted degree of the first integral related to ρ? How is the constant appearing in the first integral related to the arbitrary constant appearing in the local expansion? What are the conditions on d for the equation to have the Painlevé property? Can the local solution around the singularity analysis be expanded in Puiseux series for all P_d?

3.6 Consider the system

$$\ddot{x} = a\dot{x} + P_d(x), \tag{3.260}$$

where, as before, P_d is a polynomial of degree $d > 1$. Show that, in general, there is no Puiseux series for all values of a. Consider a simple form of $P_d = bx + x^d$ and find the conditions on a, b for the local solutions to be Puiseux series. Can you find a first integral in these cases?

3.7 For the Hénon-Heiles system (3.156), apply Painlevé test #2 for each of the the conditions listed in (3.161). Find conditions on the parameter a for the solution to be single-valued (Weiss, 1984; Tabor, 1989). For these values, find a first integral.

3.8 For the cases where the Kovalevskaya exponents of the Hénon-Heiles system (3.156) are irrational, use the construction of Section 3.11.1 to build a Kovalevskaya fractal.

3.9 Consider the planar vector field (Bountis & Tsarouhas, 1988)

$$\dot{x}_1 = -ax_1 + x_2 + x_1(x_1^2 + x_2^2), \tag{3.261.a}$$
$$\dot{x}_2 = -ax_2 - x_1 + x_2(x_1^2 + x_2^2). \tag{3.261.b}$$

Apply the singularity analysis and determine which type of movable singularities the solution may exhibit. Compare your result with the explicit solution obtained by integrating the system in polar coordinates.

3.10 **The Rikitake system**. This model reads (see Example 3.27)

$$\dot{x}_1 = -\mu x_1 - bx_2 + x_2 x_3, \tag{3.262.a}$$
$$\dot{x}_2 = -\mu x_2 + bx_1 + x_1 x_3, \tag{3.262.b}$$
$$\dot{x}_3 = -x_1 x_2 + a. \tag{3.262.c}$$

Show that the Rikitake system passes Painlevé test #2 only if (i) $a = 0$, or (ii) $b = \mu = 0$. Prove that in these two cases, the system has the Painlevé property. In the first case, it has a quadratic time-dependent first integral I. Use this first integral and the transformation $x_1 + x_2 = Iu\exp(-\mu t)$, $s = \exp(-\mu t)$ to obtain a second order equation for u. Show that this second order equation is a special case of the Painlevé equation PIII. In the second case, show that there exist two quadratic first integrals which can be used to integrate the general solutions in terms of elliptic functions.

3.11 Find conditions on the parameters a, b, c for the system

$$\dot{x}_1 = a(x_2 - x_1), \tag{3.263.a}$$
$$\dot{x}_2 = x_1(b - a - x_3) + bx_2, \tag{3.263.b}$$
$$\dot{x}_3 = x_1 x_2 - cx_3, \tag{3.263.c}$$

to pass Painlevé test #2. In each resulting case, try to integrate completely the system.

3.12 Consider the equation

$$\frac{d^4x}{dt^4} + 3x\frac{d^2x}{dt^2} - 4\left(\frac{dx}{dt}\right)^2 = 0. \tag{3.264}$$

Find both the dominant exponent and the set of Kovalevskaya exponents. Also, show that this system passes Painlevé test #2. Further show that this system does not pass Painlevé test #3, by computing the companion transformation and finding the linear eigenvalues of the companion system around each fixed point. Show that the unstable and stable manifolds around the non-vanishing fixed point can be analytically linearized whereas the fixed point cannot be linearized. Compute the normal form and show that it is not linear to order 8. Conclude that the general solution of (3.264) exhibits a movable logarithmic branch point and does not have the Painlevé property (as was first shown by Conte, Fordy, and Pickering (1993)).

3.13 The following system arises from the reduction of a semilinear parabolic PDE (Palais, 1988):

$$\dot{x}_1 = x_1(a + bx_2), \tag{3.265.a}$$
$$\dot{x}_2 = cx_1^2 + dx_2, \tag{3.265.b}$$

with $b > 0$, $c > 0$. The existence of a finite time blow-up for this reduced ODE is used to prove the existence of a finite time blow-up of some PDEs. Find all balances and prove that this system has finite time blow-up for open sets of initial conditions if and only if $bc > 0$. Show that the blow-up occurs both in the first ($\{+,+\}$) and fourth ($\{-,+\}$) quadrants. What can you say about the possibility of blow-up on one-dimensional manifolds?

3.14 Consider the one-degree-of-freedom Hamiltonian system

$$\ddot{x} = ax^d + P_d(x), \tag{3.266}$$

where $P_d(x)$ is a polynomial of degree less than d with $g(0) = 0$, $d \geq 3$ and $a \neq 0$. Show that depending on the parity of d and the sign of d, this system can exhibit both forward and backward finite time blow-up. Determine the signs of $x, \dot{x}$ when these blow-ups occur. Given that this system has a Hamiltonian $H = \frac{\dot{x}^2}{2} + V(x)$ with potential $V(x) = -a\frac{x^{d+1}}{d+1} - \int^x P_d(x)dx$, show that the blow-up time t_* can be explicitly computed in terms of the potential. That is,

$$t_* = \int_{x_0}^{\infty} \frac{dx}{\sqrt{2[E - V(x)]}}, \tag{3.267}$$

with $E = H(x_0, \dot{x}_0)$.

3.15 Find conditions on the leading exponents $\mathbf{p}$ so that the existence of a forward blow-up implies the existence of a backward blow-up.

3.16 The following system was introduced by Klapper, Rado and Tabor (1996) to model ideal three-dimensional incompressible inviscid magnetohydrodynamics. The trace equations near magnetic null points are

$$T_n = \text{Trace}\Big[\big(\nabla\mathbf{u}\big)^n\Big] \quad \text{and} \quad P_{n,m} = \text{Trace}\Big[\big(\nabla\mathbf{u}\big)^n\big(\nabla\mathbf{b}\big)^m\Big], \tag{3.268}$$

where $\mathbf{u}$ is the the fluid velocity field and $\mathbf{b}$ is the magnetic field. The significance of a finite time singularity in the magnetic field is the associated blow-up in the current, which in turn can cause the release of vast amounts of energy. This singularity is believed to be the mechanism behind the formation of solar flares. The Klapper-Rado-Tabor model gives the evolution of the gradient of the magnetic field whereas the current is given by the curl of the magnetic field. The blow-up of the gradient causes the curl to blow-up. The model, written in the variables $(x_1, x_2, x_3, x_4, x_5, x_6) \equiv (T_2, T_3, P_{1,2}, P_{2,2}, P_{1,1}, P_{2,1})$, reads

$$\dot{x_1} = -2x_2 + 2x_3, \tag{3.269.a}$$
$$\dot{x_2} = -\frac{1}{2}x_1^2 - \beta x_1 + 3x_4, \tag{3.269.b}$$
$$\dot{x_3} = \frac{1}{6}\beta^2 + \frac{\beta}{3}x_1 - x_4, \tag{3.269.c}$$
$$\dot{x_4} = -\frac{1}{3}x_1x_3 + \frac{1}{3}\beta x_3 - \frac{2}{3}\beta x_2 + \frac{2}{3}\gamma x_5, \tag{3.269.d}$$
$$\dot{x_4} = \gamma - x_6, \tag{3.269.e}$$
$$\dot{x_6} = -\frac{1}{3}x_1x_5 + \frac{1}{3}\beta x_5. \tag{3.269.f}$$

(i) Prove the existence of finite time singularities for this system. *(ii)* Give an estimate of the blow-up time. (Hint: close to the singularity, the dynamic is controlled by the most dominant part of the vector field, that is, the truncation $\mathbf{f}^{(0)}$ of the vector field associated with the given balance. For the system $\dot{\mathbf{f}} = \mathbf{f}^{(0)}$ associated with the principal balance, the two first equations decouple leading to a simple closed system $\ddot{x}_1 = x_1^2$. Therefore, if the system is close enough to the singularity, the blow-up time can be predicted by integrating this system (Goriely & Hyde, 1998)).

3.17 Use the methods of Chapter 2 to show that the Toda system with indefinite metric

$$\dot{a}_i = s_{i+1} b_i^2 - s_{i-1} b_{i-1}^2, \qquad i = 1, \dots, N, \tag{3.270.a}$$

$$\dot{b}_i = \frac{1}{2} b_i (s_{i+1} a_{i+1} - s_i a_i), \qquad i = 1 \dots, N-1, \tag{3.270.b}$$

has a Lax pair A, B ($\dot{A} = [B, A]$) with A, a tridiagonal matrix. Find the corresponding first integrals and recover the conditions on the signs s_i for the absence of finite time blow-up.

Chapter 4:

Polynomial and quasi-polynomial vector fields

"A system of differential equations is only more or less integrable."
Poincaré

Linear systems of n first order ordinary differential equations with constant coefficients can be written in the matrix form

$$S(M;\mathbf{x}) : \dot{\mathbf{x}} = M\mathbf{x}, \tag{4.1}$$

where $\mathbf{x} \in \mathbb{C}^n$ and $M \in \mathrm{M}_n$ ($\mathrm{M}_{n,m}(\mathbb{K})$ is the set of $(n \times m)$ matrices over a field $\mathbb{K}$ and $\mathrm{M}_n = \mathrm{M}_{n\times n}(\mathbb{K})$). This matrix formalism for linear systems is naturally related to $\mathrm{GL}(n,\mathbb{C})$ the group of invertible linear transformations since the linear transformation $L_C : \mathbf{x} = C\mathbf{x}'$ with $C \in \mathrm{GL}(n,\mathbb{C})$ defines an involution in the set of linear systems $\mathrm{LS} = \{S(M;\mathbf{x}), M \in \mathrm{M}_n\}$ such that

$$L_C(S(M;\mathbf{x})) = S(C^{-1}MC;\mathbf{x}'). \tag{4.2}$$

It is a standard matter to use this equivalence relation to define equivalence classes and a canonical form for linear systems of ODEs from which most properties can be readily found. The canonical form is defined by the Jordan form of matrix M. Most of the properties of the solutions can be readily associated with the properties of matrix M. For instance, the existence of a linear first integral in the variables $\mathbf{x}$ results from the degeneracy of matrix M.

Proposition 4.1 *The system $S(M,\mathbf{x})$ admits a linear first integral $I = \mathbf{a}.\mathbf{x}$ if and only if $\det(M) = 0$.*

Proof. If $I = \mathbf{a}.\mathbf{x}$ is a first integral, then $\dot{I} = \mathbf{a}.\dot{\mathbf{x}} = \mathbf{a}.(M\mathbf{x}) = 0$. This must be true for all $\mathbf{x}$, hence $\mathbf{a}M = 0$ and M admits a non empty kernel. Hence, $\det(M) = 0$. Now, let M be such that $\det(M) = 0$. Therefore, there exists an $\mathbf{a} \neq \mathbf{0}$ such that $\mathbf{a}M = 0$ and $\mathbf{a}.\mathbf{x}$ is a first integral of the system. □

Linear systems play a central role in the theory of dynamical systems, and some of the most important properties to prove nonintegrability can be obtained from their analysis (see Section 5.1). In this chapter, we develop a similar approach for nonlinear vector fields, the so-called *quasimonomial (QM) formalism*. The QM formalism is a description of vector fields based on the algebraic structure of a class of functions which generalizes the polynomial functions in many variables. This class of functions can be used to define the QM-class, the set of systems of first order nonlinear ODEs whose vector field are sums of quasimonomial terms. The construction follows by defining nonlinear transformations acting on the systems of ODEs in the QM-class. Both the QM-systems and the nonlinear transformations have a representation in terms of matrices. This allows us to reveal certain natural group structures for nonlinear systems of ODEs and divide the set of QM-systems into equivalence classes. Finally, a canonical form can be built for each equivalence class.

The construction of canonical forms can be visualized as a nonlinear analog of the construction of systems of linear differential equations based on the isomorphism between the space of solutions of $S(M)$ and $\mathbb{R}^n$. We show how, for nonlinear systems of ODEs, the analysis can be carried out on the canonical form. The description that follows is based on the papers by Brenig and Goriely (Brenig & Goriely, 1989; Goriely, 1992) but we give here the foundations of the formalism in a unified way by focusing on the algebraic structure of vector fields and the corresponding geometric construction in the space of exponents.

4.1 The quasimonomial systems

Let $\mathbb{K}$ be a field of constants and $\Sigma = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. A *quasimonomial* q over $\mathbb{K}$ is defined as $q = \mathbf{x}^{\mathbf{b}} = \prod_{i=1}^{n} x_i^{b_i}$, $b_i \in \mathbb{K}$, $\mathbf{x} \in \mathbb{C}^n$. A *quasimonomial function* is a finite sum of quasimonomials $f : \mathbb{C} \to \Sigma : \mathbf{x} \to \sum a_i \prod_{j=1}^{n} x_j^{b_{ij}}$. By extension we define the class $\mathcal{Q}_n$.

Definition 4.1 The set $\mathcal{Q}_n$ of *quasimonomial systems in n dimensions* is the set of systems of n first order ODEs whose vector fields are quasimonomial functions. That is, a system S is in $\mathcal{Q}_n$ if it is of the form

$$S : \dot{x}_i = x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}, \qquad i = 1, \ldots, n, \tag{4.3}$$

where $A \in \mathrm{M}_{n,m}(\mathbb{C})$, $B \in \mathrm{M}_{m,n}(\mathbb{K})$.

The quasimonomial class can easily be generalized to include formal series with infinite sums of quasimonomials. To do so, we consider two sequences of vectors $A = \{\mathbf{A}_i,\ i = 1, \ldots, \infty\}$ and $B = \{\mathbf{B}_i,\ i = 1, \ldots, \infty\}$ and build a similar vector field. In particular, this allows us to include all analytic vector fields and formal power series. In this case, most of the results presented in the next sections remain true (with appropriate changes in the notation). However, for the sake of simplicity, here we only discuss the finite case.

System (4.3) can be written in the symbolic form $S(A, B; \mathbf{x}, t)$ which stresses the role of the matrices A and B in the underlying group structure. Typically, we think of $\mathbb{K}$ as being $\mathbb{Z}$, $\mathbb{N}$ or $\mathbb{Q}$. For instance, the set of n first order ODEs with polynomial vector fields (denoted hereafter $\mathcal{P}_n$) is a subset of $\mathcal{Q}_n$ for which $\mathbb{K} = \mathbb{N} \cup \{-1\}$ and A is chosen accordingly. An ODE belonging to the QM class

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{4.4}$$

is uniquely determined by the matrices A and B with the following convention: $\mathbf{B}_{1.} = \mathbf{0}$ if $\mathbf{A}_{.1} \neq \mathbf{0}$, and $\mathbf{A}_{.i} \neq \mathbf{0}\ \forall\ i$ (no zero line vectors). The others quasimonomials are ordered according to the lexicographic order and the condition $\mathbf{B}_{i.} \neq \mathbf{B}_{j.} \forall\ i \neq j$. With this convention, we can write unequivocally $S(A, B; \mathbf{x}, t) = S(\mathbf{f}; \mathbf{x}, t)$ for any quasimonomial function $\mathbf{f}$. The *degree* d of a quasimonomial $\mathbf{x}^{\mathbf{B}_{i.}}$ corresponding to the vector $\mathbf{B}_{i.} \in \mathbb{K}^n$ is defined as $|\mathbf{B}_{i.}| = \sum_{j=1}^{n} B_{ij} = d - 1$. As a consequence, there is a one-to-one relation between the QM class and the space of couples of matrices $(A, B) \in (\mathrm{M}_{n,m}(\mathbb{C}), \mathrm{M}_{n,m}(\mathbb{K}))$ defined by Equation (4.3).

Example 4.1 **A planar system.** Consider the two-dimensional vector field

$$\dot{x}_1 = 5x_1^2 - 3x_1x_2^2, \tag{4.5.a}$$
$$\dot{x}_2 = 4x_1 + x_1x_2 - 6x_2^2. \tag{4.5.b}$$

Then, the corresponding matrices A and B are

$$A = \begin{bmatrix} -3 & 0 & 5 \\ 6 & 4 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4.6}$$

■

4.2 The quasimonomial transformations

The advantage of this representation of vector fields becomes clear when a particular set of nonlinear transformations is considered.

Definition 4.2 A *quasimonomial transformation* (QMT) $T : \Sigma^n \to \Sigma^n$ is a product over the independent variables

$$T_C : x_i = \prod_{k=1}^{n} {x'}_k^{C_{ik}}, \qquad i = 1, \ldots, n, \tag{4.7}$$

where $C \in \mathrm{GL}(n, \mathbb{K})$.

In general, the inverse transformation cannot be uniquely defined on Σ^n and the appropriate covering surface should be introduced. The multi-valuedness of the Riemann surface associated with the inverse transformation has to be taken into account. That is,

$$T_C^{-1} : \Sigma^n \to \Sigma^n : x'_i = \prod_{k=1}^{n} x_k^{C_{ik}^{-1}} \exp(2\pi m_i \alpha_k), \qquad i = 1, \dots, n, \tag{4.8}$$

with $\alpha_k = \sum_{k=1}^{n} C_{ik}^{-1}$ and $m_i \in \mathbb{Z}$. The *principal branch* of the inverse transformation is defined by $m_i = 0 \; \forall \; i$. There are two important cases where the inverse transformation is unique. In the first case we restrict the domain of existence of the dependent variables, and in the second case we restrict the set of matrices C.

Proposition 4.2 *Let $(\mathbb{R}_0^+)^n$ be the first strictly positive orthant of $\mathbb{R}^n$ and assume that $\mathbf{x}, \mathbf{x}' \in (\mathbb{R}_0^+)^n$. Then, the QMT forms a group for the composition law $T_{C_1} \circ T_{C_2} = T_{(C_1 C_2)}$.*

Proposition 4.3 *Let $\mathbb{K} = \mathbb{Z}$ and assume that C belongs to the unimodular group, that is, the matrices in* $\mathrm{GL}(n, \mathbb{Z})$ *with determinants equal to ± 1. Then, the QMT forms a group for the composition law $T_{C_1} \circ T_{C_2} = T_{C_1 C_2}$.*

In these two cases, the inverse transformation is uniquely defined and reads

$$T_C^{-1} = T_{C^{-1}}, \qquad i = 1, \dots, n. \tag{4.9}$$

We now study the transformation of QM systems under the action of a QMT. By extension, the QMTs act on the QM class. The following proposition states that these transformations leave the QM systems form-invariant.

Proposition 4.4 *Let $C \in \mathrm{GL}(n)$, then $T_C : \mathcal{Q}_n \to \mathcal{Q}_n$ is an involution of $\mathcal{Q}_n$.*

Proof. When applied to a QM system (4.3), the transformation (4.7) gives

$$S' : \dot{x}'_i = x'_i \sum_{j=1}^{m} A'_{ij} \prod_{k=1}^{n} {x'}_k^{B'_{jk}}, \qquad i = 1, \dots, n, \tag{4.10}$$

where

$$A' = C^{-1} A, \quad B' = BC. \tag{4.11}$$

That is,

$$T_C(S(A, B; \mathbf{x}, t)) = S(A', B'; \mathbf{x}', t). \tag{4.12}$$

□

Example 4.2 Consider the planar vector field (4.5) from the previous example. The transformation $x'_1 = x_1^2 x_2,\; x'_2 = x_1 x_2$ is a quasimonomial transformation T_C with matrix $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. The new system $S(A', B')$ is defined by the matrices

$$A' = C^{-1} A = \begin{bmatrix} -9 & -4 & 4 \\ 15 & 8 & -3 \end{bmatrix}, \qquad B' = BC = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}. \tag{4.13}$$

That is, the new system reads

$$\dot{x}'_1 = x'_1(-9{x'_1}^2 {x'_2}^2 - 4x'_1 + 4{x'}_1^2 x'_2), \tag{4.14.a}$$
$$\dot{x}'_2 = x'_2(15{x'_1}^2 {x'_2}^2 + 8x'_1 - 3{x'}_1^2 x'_2). \tag{4.14.b}$$

■

An equivalence relation between vector fields in the QM class can be defined with respect to the quasimonomial transformation.

Definition 4.3 Two systems $S_1, S_2 \in \mathcal{Q}_n$ are *$\mathcal{Q}$-equivalent*, if there exists $C \in \mathrm{GL}(n, \mathbb{C})$ such that $T_C(S_1) = S_2$.

This equivalence relation can be easily tested on the matrices (A, B). Two matrices $M, N \in \mathrm{M}_{m,m}$ are *equivalent* $M \cong N$ if there exists a permutation σ of the indices $\{1, 2, \dots, m\}$, such that $M_{i,j} = M_{\sigma(i),\sigma(j)}$. That is, two matrices are equivalent if they are equal after a simultaneous permutation of rows and columns.

Proposition 4.5 *Two systems $S_1(A_1, B_1)$, $S_2(A_2, B_2) \in \mathcal{Q}_n$ are $\mathcal{Q}$-equivalent if and only if $B_1 A_1 \cong B_2 A_2$.*

Proof. If $B_1 A_1 \cong B_2 A_2$, there exists an invertible matrix C such that $B_2 = B_1 C$ and $A_2 = C^{-1} A_1$. Conversely, if S_1 and S_2 are $\mathcal{Q}$-equivalent, then there exists an invertible matrix C and, according to the transformation rules (4.11), $B_2 = B_1 C$ and $A_2 = C^{-1} A_1$. The identity follows. □

At this stage, the notion of $\mathcal{Q}$-equivalence defined here is a formal equivalence between vector fields. Nevertheless, it can be related to other notions of equivalence (Guckenheimer & Holmes, 1983, p. 38).

Definition 4.4 Two vector fields $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$ are *C^k-equivalent* if there exists a C^k diffeomorphism which takes the orbits of $\mathbf{f}$ to the orbits of $\mathbf{g}$ preserving the direction of the orbits.

If the diffeomorphism preserves the parametrization in time, the vector fields are *C^k-conjugate* (*topologically conjugate* in the case $k = 0$). In general, $\mathcal{Q}$-equivalence between two vector fields does not imply C^k-equivalence. However, in the particular case where the assumptions of Proposition 4.2 are satisfied, the quasimonomial transformation is a diffeomorphism.

Proposition 4.6 *Consider $S : \dot{\mathbf{x}} = f(\mathbf{x}) \in \mathcal{Q}_n$ and let $C \in \mathrm{GL}(n)$. If $\mathbf{x}(t) \in (\mathbb{R}_0^+)^n \ \forall\ i = 1, \dots, n \ \forall\ t \in \mathbb{R}$, then there exists $k \in \mathbb{N}_0$ such that $T_C(S)$ is C^k-equivalent to S.*

4.3 New-time transformations

So far, we have considered transformations acting on the dependent variables $\mathbf{x}$. To obtain a complete set of nonlinear transformations, we also include transformations acting on the independent variables t. Since t does not appear explicitly in the QM-systems, we introduce an implicit time-parametrization acting on the differential dt.

Definition 4.5 A *new-time transformation* (NTT) $N_{\boldsymbol{\beta}} : \mathbb{C} \to \mathbb{C}$ is defined as

$$N_{\boldsymbol{\beta}} : dt = \prod_{i=1}^{n} x_i^{\beta_i} d\tilde{t}. \tag{4.15}$$

When needed, the explicit time-transformation can be found by integrating both sides to obtain

$$(t - t_0) = \int_{\tilde{t}_0}^{\tilde{t}} \prod_{i=1}^{n} x_i^{\beta_i}(t') dt'. \tag{4.16}$$

By extension, we consider the action of the NTT on a system $S \in \mathcal{Q}_n$ and note that the QM class is an involution under the new-time transformation. That is,

$$N_{\boldsymbol{\beta}} : \mathcal{Q}_n \to \mathcal{Q}_n : N_{\boldsymbol{\beta}}(S(A, b; \mathbf{x}, t)) = S(A, B \oplus \boldsymbol{\beta}; \mathbf{x}, \tilde{t}), \tag{4.17}$$

where $(B \oplus \boldsymbol{\beta})_{ij} = B_{ij} + \beta_j$.

By analogy with the QMT, an equivalence relation based on the NTT can also be derived. However, this equivalence does not play an important role in the latter and is left as an exercise.

Example 4.3 Consider the planar vector field (4.5) from the two previous examples together with the NTT $dt = x_1' d\tilde{t}$. That is, $\boldsymbol{\beta} = (-1, 0)$. The new system is defined by the matrices $\tilde{A}' = A'$ and $\tilde{B}' = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$. That is,

$$\frac{dx_1'}{d\tilde{t}} = x_1'(-9x_1'{x_2'}^2 - 4 + 4x_1'x_2'), \tag{4.18.a}$$

$$\frac{dx_2'}{d\tilde{t}} = x_2'(15x_1'{x_2'}^2 + 8 - 3x_1'x_2'). \tag{4.18.b}$$

■

4.4 Canonical forms

The $\mathcal{Q}$-equivalence divides the QM class into equivalence classes. In turn, each class is formed by the set of systems $S(A, B) \in \mathcal{Q}_n$ with equal matrix $M = BA$. Therefore, each equivalence class is characterized by one *canonical form* which generates the whole equivalence class by application of quasimonomial transformations. In order to define these canonical forms we consider two cases.

- **Case 1:** $m = n$ **and** $B \in \mathrm{GL}(n)$**.** Since B is invertible, it can be used to define a QMT for any system $S(A, B)$ in a given equivalence class. That is, we choose

$$T_{B^{-1}}(S(A, B; \mathbf{x}, t)) = S_{LV}(M, \mathrm{I}; \mathbf{x}', t). \tag{4.19}$$

  The *canonical form* S_{LV} defines a system of ODEs with only quadratic nonlinearities, the so-called *Lotka-Volterra* system (Lotka, 1956)

$$S_{LV}(M, \mathrm{I}; \mathbf{x}, t) : \dot{x}_i = x_i \sum_{j=1}^{n} M_{ij} x_j, \qquad j = 1, \ldots, n. \tag{4.20}$$

  That is,

$$S_{LV}(M, \mathrm{I}; \mathbf{x}, t) : \dot{\mathbf{x}} = \mathbf{x}(M\mathbf{x}). \tag{4.21}$$

- **Case 2:** $m \neq n$ **or** $\det(B) = 0$. In this case it is possible to find a suitable embedding of the $\mathbf{x}$ variables such that in the new variables the new system satisfies the assumptions of Case 1 (see Brenig and Goriely (1989) for a complete description). We first describe the embedding. Consider a system $S(A, B; \mathbf{x}, t) \in \mathcal{Q}_n$ and assume that $\mathrm{rank}(B) = n$. The system S is an n-dimensional system. It can be formally embedded in an m-dimensional system $S(\widehat{A}, \widehat{B}; \widehat{\mathbf{x}}, t)$ with $\mathbf{x} \in \mathbb{C}^m$ and

$$\widehat{A} = \begin{bmatrix} A \\ 0 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} B & b \end{bmatrix}, \tag{4.22}$$

  where $\widehat{A}$, $\widehat{B} \in \mathrm{M}_m$ and $b \in \mathrm{M}_{m-n,n}$ is chosen in such a way that $\widehat{B}$ is invertible. The m-dimensional system $S(\widehat{A}, \widehat{B}; \widehat{\mathbf{x}}, t)$ is identical to $S(A, B; \mathbf{x}, t)$ if the following initial conditions are imposed on the embedding variables:

$$x_i(t_0) = 1 \qquad i = n + 1, \ldots, m. \tag{4.23}$$

  The new system $S(\widehat{A}, \widehat{B})$ satisfies the assumptions of Case 1 and the canonical form can be defined by choosing

$$T_{\widehat{B}^{-1}}(S(\widehat{A}, \widehat{B}; \mathbf{x}, t)) = S_{LV}(M, \mathrm{I}; x', t), \tag{4.24}$$

with $M = \widehat{B}\widehat{A} = BA$. If B is of rank $r < n$ then it is still possible to find the Lotka-Volterra canonical form by the same procedure (Goriely & Brenig, 1990). This property results from the fact that if one of the matrices A or B is of rank $r < n$, then $S(A, B)$ is *algebraically degenerate* and can be reduced to a r-dimensional system of the same form but with different matrices A and B (see Section 4.8).

The Lotka-Volterra system appears to play a central role in the theory of nonlinear vector fields since most analysis can be performed on the canonical form $S_{LV}(M, \mathrm{I})$ rather than on the original system $S(A, B)$. Consequently, the main properties of a system depend only on the matrix M rather than on the particular nonlinear structure determined by the matrices A and B. Before considering these aspects in greater detail, we introduce some geometric tools.

4.5 The Newton polyhedron

Any given system in the quasimonomial class is uniquely characterized by two matrices (A, B). Matrix B describes the nonlinearities of the system. There is an elegant geometric construction associated with this set. Each row of B can be represented by a point in $\mathbb{K}^n$, the *space of exponents*. A similar construction is given in Bruno (1989). Only relevant definitions are included here, a more thorough description of the Newton polyhedron can be found in Buseman (1958).

Let $\mathbb{K}$ be a field of constants and consider in $\mathbb{K}^n$ a finite set of points $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$, $\mathbf{b}_i \in \mathbb{K}^n \ \forall\ i$.

Definition 4.6 Let $\mathbf{v}$ be a fixed point in $\mathbb{K}^n$ and $c = \max(B\mathbf{v})$, the *support hyperplane* $L_{\mathbf{v}}$ of B and the *supporting half-space* $L_{\mathbf{v}}^{(-)}$ are defined by the set of points $\mathbf{X} \in \mathbb{R}^n$ such that

$$L_{\mathbf{v}} : \mathbf{X}\mathbf{v} = c, \tag{4.25.a}$$
$$L_{\mathbf{v}}^{(-)} : \mathbf{X}\mathbf{v} \le c. \tag{4.25.b}$$

Definition 4.7 The *Newton polyhedron* Γ is the intersection of all supporting half-spaces $L_{\mathbf{v}}^{(-)}$ of B. That is,

$$\Gamma = \bigcap_{\mathbf{v}\neq 0} L_{\mathbf{v}}^{(-)} \qquad \forall\ \mathbf{v} \in \mathbb{K}^n. \tag{4.26}$$

The boundary $\partial\Gamma$ is called the *Newton polyhedron*. Since B is a finite set of points, we have $\partial\Gamma = \Gamma$. Moreover, the Newton polyhedron Γ coincides with the closure of the convex hull of B (that is, the smallest convex set containing B) (Buseman, 1958; Goldman & Tucker, 1956).

A *face* of Γ is the intersection of Γ with $L_{\mathbf{v}}$. The Newton polyhedron is composed by the faces of Γ. Since B is a finite set, Γ is a polyhedron whose faces are hyperplanes of dimensions $d = 1, \ldots, n-1$. Let $\Gamma_j^{(d)}$ be the d-dimensional faces of Γ. Similarly, the subset $B_j^{(d)} = \Gamma_j^{(d)} \cap B$ is composed by the points of B lying on the faces $\Gamma_j^{(d)}$. The Newton polyhedron has the following interesting property. Let $\tilde{\mathbf{b}} \in L_{\mathbf{v}}$, then

$$(\mathbf{b}, \mathbf{v}) = (\tilde{\mathbf{b}}, \mathbf{v}) \quad \forall\ \mathbf{b} \in \Gamma_j^{(d)}, \tag{4.27.a}$$
$$(\mathbf{b}, \mathbf{v}) < (\tilde{\mathbf{b}}, \mathbf{v}) \quad \forall\ \mathbf{b} \in \Gamma\backslash\Gamma_j^{(d)}. \tag{4.27.b}$$

The first equation states that $\Gamma_j^{(d)} \subset L_{\mathbf{v}}$ while the second ensures that $L_{\mathbf{v}}$ is a supporting half-space.

Example 4.4 Newton's polygon. Let $\mathbb{K} = \mathbb{Z}$, $n = 2$ and

$$\begin{aligned} B &= \{\{0,2\},\{2,3\},\{3,1\},\{2,2\}\}, \\ &= \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}. \end{aligned} \tag{4.28}$$

The Newton polygon is the triangle of vertices $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ (see Figure 4.1). The faces $\Gamma^{(0)}$ are the vertices $\Gamma_j^{(0)} = \mathbf{b}_j$ $(j = 1, 2, 3)$; the faces $\Gamma_j^{(1)}$ are the line segments connecting $\mathbf{b}_j$ to $\mathbf{b}_{j+1 (\mathrm{mod}\ 3)}$ $(j = 1, 2, 3)$ and the

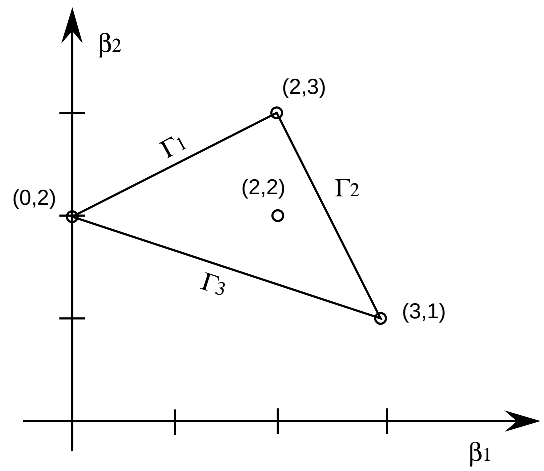

Figure 4.1: The Newton polygon of the set B.

subsets $B_j^{(d)}$ are given by

$$B_j^{(0)} = \mathbf{b}_j, \qquad j = 1, 2, 3, \tag{4.29.a}$$

$$B_j^{(1)} = \{\mathbf{b}_j, \mathbf{b}_{j+1(\mathrm{mod}\ 3)}\}, \qquad j = 1, 2, 3. \tag{4.29.b}$$

■

4.6 Transformation of the Newton polyhedron

By extension, a Newton polyhedron can be associated with each vector field in $\mathcal{Q}_n$.

Definition 4.8 The *Newton polyhedron* of $S(A, B; \mathbf{x}, t) \in \mathcal{Q}_n$ is the Newton polyhedron of the set defined by the rows of B.

Example 4.5 A general Lotka-Volterra system. Consider the general 3D Lotka-Volterra vector field with a linear diagonal part and a constant term

$$\dot{x}_i = \alpha_i + \lambda_i x_i + x_i \sum_{j=1,2,3} M_{ij} x_j, \qquad i = 1, 2, 3. \tag{4.30}$$

The corresponding matrices A and B are

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} \alpha_1 & 0 & 0 & \lambda_1 & M_{11} & M_{112} & M_{13} \\ 0 & \alpha_2 & 0 & \lambda_2 & M_{21} & M_{22} & M_{23} \\ 0 & 0 & \alpha_3 & \lambda_2 & M_{31} & M_{32} & M_{33} \end{bmatrix}. \tag{4.31}$$

The Newton polyhedron of this system is the octahedron built on the 6 non-zero row vectors of B (see Figure 4.2). ■

In a change of variables, the vector field is transformed into a new vector field with a new Newton polyhedron. In the particular case of the quasimonomial transformations, the transformation of the Newton polyhedron is

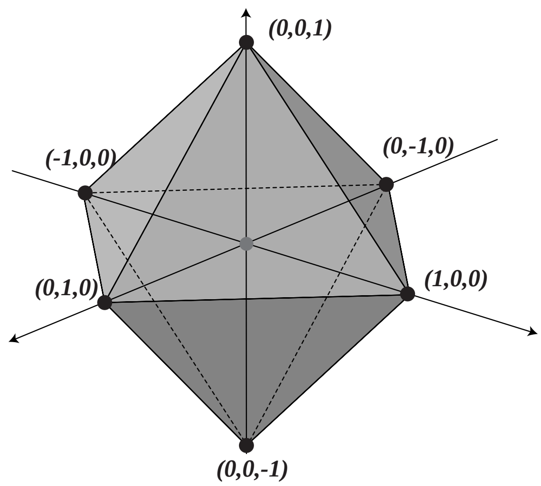

Figure 4.2: The Newton polyhedron associated with the 3D Lotka-Volterra system is an octahedron.

particularly simple. For a given system $S(A, B)$ and a given transformation T_C, the new polyhedron can be obtained from the old one.

Proposition 4.7 *In a quasimonomial transformation T_C, the Newton polyhedron of the system $S(A, B; \mathbf{x}, t)$ is transformed onto the Newton polyhedron of the system $T_C(S) = S(A', B'; \mathbf{x}', t)$ by the linear transformation*

$$\Gamma' = \Gamma C, \tag{4.32.a}$$

$$\Gamma'^{(d)}_j = \Gamma^{(d)}_j C \qquad \forall\ j, d, \tag{4.32.b}$$

$$B'^{(d)}_j = B^{(d)}_j C \qquad \forall\ j, d. \tag{4.32.c}$$

Proof. The quasimonomial transformation T_C acts in the space of exponents $\mathbb{K}^n$ as a linear transformation mapping the set of points B onto the set $B' = BC$. The convex hull of the set B is defined by the linear equalities (4.27). These relations are preserved by the linear transformation induced by the QMT. Therefore the polyhedron Γ is transformed by a linear transformation. □

However, in general, nonlinear transformations acting on a set of points B in the space of exponents $\mathbb{K}^n$ do not preserve the polyhedron Γ. Indeed, some points of B which are not on the faces of Γ may be transformed in a nonlinear transformation on vertices of Γ'. The same types of results hold for the new-time transformation.

Proposition 4.8 *The transformation $N_{\boldsymbol{\beta}}$ maps the Newton polyhedron of the system $S(A, B; \mathbf{x}, t)$ onto the Newton polyhedron of the set $N_{\boldsymbol{\beta}}(S) = S(A, \widetilde{B}; \mathbf{x}, \widetilde{t})$ by the linear transformation:*

$$\widetilde{\Gamma} = \Gamma + \boldsymbol{\beta}, \tag{4.33.a}$$

$$\widetilde{\Gamma}^{(d)}_j = \Gamma^{(d)}_j + \boldsymbol{\beta} \qquad \forall\ j, d, \tag{4.33.b}$$

$$\widetilde{B}^{(d)}_j = B^{(d)}_j + \boldsymbol{\beta} \qquad \forall\ j, d. \tag{4.33.c}$$

The new-time transformation acts in the space of exponents as a parallel transformation. The proof can be easily adapted from the previous proposition.

Example 4.6 Consider again the set (4.28) with Newton's polygon shown in Figure 4.1. We apply a transformation T_C with matrix C given by

$$C = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}. \tag{4.34}$$

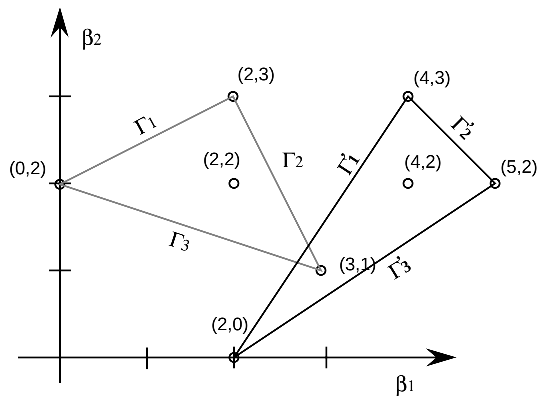

Figure 4.3: The Newton polygon of the set $B' = BC$.

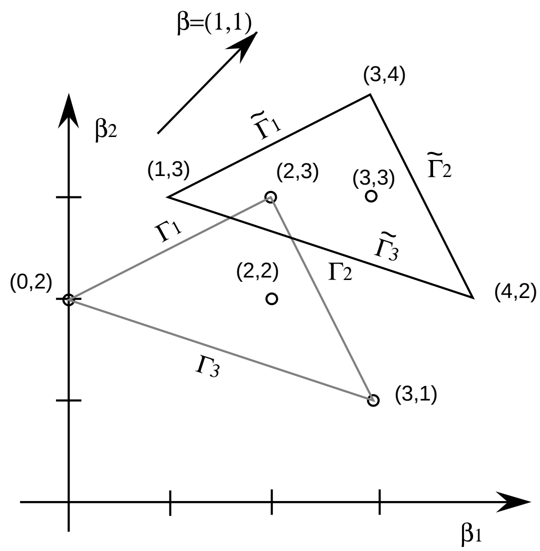

Figure 4.4: The Newton polygon of the set $\widetilde{B} = B \oplus \boldsymbol{\beta}$.

The Newton polygon of the set $B' = BC$ is shown in Figure 4.3. Now, consider the vector $\boldsymbol{\beta} = (1, 1)$ and the corresponding new-time transformation

$$\widetilde{B} = B \oplus \boldsymbol{\beta} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \\ 4 & 2 \\ 3 & 3 \end{bmatrix}. \tag{4.35}$$

The corresponding Newton polygon is shown in Figure 4.4. ■

4.7 Historical digression: a new-old formalism

The quasimonomial formalism is not a new representation for differential equations. It seems to have been re-discovered independently at least four times. The more recent formulation dates from 1987-93 and is due to Brenig and Goriely (Brenig, 1988; Brenig & Goriely, 1989; Goriely, 1992; Brenig & Goriely, 1994).

In 1987 Goldman "introduced" the *multinomial formalism* for the search of first integrals (Goldman, 1987; Sit, 1989) in a spirit very close to the one developed by Goriely and Brenig (1990; 1992). His algorithm is a test for the existence of first integrals as quasimonomial functions with a fixed number of quasimonomials.

In 1983, Peschel and Mende developed the theory of *multinomial transformations* in the theory of biological evolution and growth processes. Their multinomial transformations are the quasimonomial transformations given here.

As early as 1963, Bruno probably was the first to introduce *power transformations*. In a series of impressive papers, he built a comprehensive theory of normal forms based on the quasimonomial formalism. He was able to obtain the optimal results on the convergence and divergence of Poincaré transformations (Poincaré transformations are the near-identity transformations used to map a given system of ODEs to its normal form). He also showed how to use power transformations to find the zeroes of analytic functions in many variables (Bruno, 1989; Bruno, 1990; Bruno, 1999).

In 1902, Weierstrass developed a method for the resolution of an analytic function. To do so, he used a finite succession of unimodular monomial transformations to find all of the possible branches of analytic curves (Bruno, 1989, p. 62). This so-called σ-*process*, is used in algebraic geometry for the resolution of singularities of algebraic curves. Bendixson used Weierstrass' method for the resolution of the singularities (in phase space) of an ODEs. The σ-process is still a useful tool in dynamical system theory to "blow-up" degenerate fixed points (Arnold, 1988a; Koppel & Howard, 1973; Jones *et al.*, 1990).

The Newton polygon also has a long history in mathematics. First introduced by Newton to solve algebraic functions in many variables, it turned out to be an important tool in many problems. For linear differential equations, the Puiseux diagram is used to study the leading behavior of Puiseux series at the singularity and was introduced by Briot and Bouquet (1856) (Fine, 1889; Ince, 1956, p. 298). In modern times, it has been used by Malgrange to study irregular singular points of ODEs (the *Newton-Malgrange polygon*) (Malgrange, 1989, 1991).

More recently, renewed interest in this formalism has led to a myriad of applications in mathematical biology and new algorithms for computing first and second integrals (Fairén & Hernández-Bermejo, 1996; Gouzé, 1993).

We show here that this unifying formalism is also the natural framework for a complete analysis of the integrability properties of systems of nonlinear ODEs.

4.8 Algebraic Degeneracy

As a first application of the quasimonomial formalism, we study the nonlinear equivalent of Proposition 4.1. We show that, if A or B is of rank $r < n$, interesting information concerning the integrability of the quasimonomial system $S(A, B; \mathbf{x}, t)$ can be obtained. First, we introduce two auxiliary matrices in order to exclude the linear diagonal terms. Let $c = \sum_{k=1}^{n} |B_{1k}|$, that is, $c = 0$ if $\mathbf{B}_{1.} = \mathbf{0}$, and define

$$\widehat{A} = \begin{cases} A & \text{if } c \neq 0, \\ A \backslash \mathbf{A}_{.1} & \text{if } c = 0, \end{cases} \qquad \widehat{B} = \begin{cases} B & \text{if } c \neq 0, \\ B \backslash \mathbf{B}_{1.} & \text{if } c = 0. \end{cases} \tag{4.36}$$

The distinction between matrices A and $\widehat{A}$ is introduced to extract the linear diagonal terms from matrices A and B. Using matrices $\widehat{A}$ and $\widehat{B}$ we can write $S(A, B; \mathbf{x}, t)$ as

$$\widehat{S}(\widehat{A}, \widehat{B}, \boldsymbol{\lambda}) : \dot{x}_i = x_i \left(\lambda_i + \sum_{j=1}^{m} \widehat{A}_{ij} \prod_{k=1}^{n} x_k^{\widehat{B}_{jk}} \right), \qquad i = 1, \ldots, n, \tag{4.37}$$

where $\boldsymbol{\lambda} = \mathbf{A}_{.1}$ if $\mathbf{B}_{1.} = \mathbf{0}$ and $\boldsymbol{\lambda} = \mathbf{0}$ otherwise.

Whenever matrix $\widehat{A}$ or matrix $\widehat{B}$ is of rank $r < n$, system $S(A, B)$ will be referred to as *algebraically degenerate*. We now prove two general results concerning algebraically degenerate systems (see also Planck (1996b; 1996a) for similar results for systems with a Poisson structure).

4.8.1 Degeneracy of matrix $\widehat{A}$

Proposition 4.9 *Consider the system $S(A,B;\mathbf{x},t) \in \mathcal{Q}_n$. The matrix $\widehat{A}$ is of rank $r < n$ if and only if $S(A,B)$ has $(n-r)$ independent first integrals of the form*

$$I = e^{(\mathbf{m}.\boldsymbol{\lambda})t} \prod_{k=1}^{n} x_k^{m_k}, \qquad \mathbf{m} \in \ker(\widehat{A}^T). \tag{4.38}$$

Proof. Assume without loss of generality that $\boldsymbol{\lambda} \neq 0$ and assume that $\text{rank}(\widehat{A}) = r < n$. Let $\mathbf{m}$ be a vector of $\ker(\widehat{A}^T)$ and consider $y = \mathbf{x}^{\mathbf{m}}$. Taking into account the transformation rule (4.11) we have

$$\dot{y} = (\mathbf{m}.\boldsymbol{\lambda})y. \tag{4.39}$$

This last relation and the existence of $(n-r)$ independent vectors in $\ker(\widehat{A}^T)$ imply the existence of $(n-r)$ independent first integrals of the type (4.38). Conversely, assume that there exist $(n-r)$ independent first integrals of the form (4.38) for a given system $S(A,B)$ with exponents $\{\mathbf{m}^{(1)},\dots,\mathbf{m}^{(n-r)}\}$. Let $C = \left[\mathbf{m}^{(1)},\dots,\mathbf{m}^{(n-r)},\mathbf{e}^{(n-r+1)},\dots,\mathbf{e}^{(n)}\right]^{\mathrm{T}}$ be an $n\times n$ matrix constructed on the exponents of the first integrals where the vectors $\{\mathbf{e}^{(n-r+1)},\dots,\mathbf{e}^{(n)}\}$ are chosen such that $C \in \mathrm{GL}(n)$. The transformation $T_C(S(A,B)) = S(A',B')$ is such that the matrix $\widehat{A}'$ constructed on A' according to (4.36), reads $\widehat{A}' = C^{-1}\widehat{A} = \begin{bmatrix} 0 \\ a \end{bmatrix}$ where $a \in \mathrm{M}_{r\times n}$ is of rank r. Since $\text{rank}(C^{-1}\widehat{A}) = \text{rank}(\widehat{A})$, the result follows. □

Example 4.7 Consider the three-dimensional Lotka-Volterra system

$$\dot{x}_1 = \lambda_1 x_1 + x_1(cx_2 + x_3), \tag{4.40.a}$$
$$\dot{x}_2 = \lambda_2 x_2 + x_2(x_1 + ax_3), \tag{4.40.b}$$
$$\dot{x}_3 = \lambda_3 x_3 + x_3(bx_1 + x_2). \tag{4.40.c}$$

Matrices A and B are given by

$$A = \begin{bmatrix} \lambda_1 & 0 & c & 1 \\ \lambda_2 & 1 & 0 & a \\ \lambda_3 & b & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.41}$$

The new matrices $\widehat{A}$, $\widehat{B}$, and λ are

$$\widehat{A} = \begin{bmatrix} 0 & c & 1 \\ 1 & 0 & a \\ b & 1 & 0 \end{bmatrix}, \qquad \widehat{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}. \tag{4.42}$$

A straightforward application of Proposition 4.9 to matrix $\widehat{A}$ leads to the condition $abc + 1 = 0$ (that is, $\det(\widehat{A}) = 0$), for which a first integral is

$$I = e^{(ab\lambda_1 - b\lambda_2 + \lambda_3)t} x_1^{ab} x_2^{-b} x_3. \tag{4.43}$$

Further, artificially, we introduce three new variables $x_{3+i} = e^{x_i}$. The new system, S_2, is now a six-dimensional system $S(\widehat{A}_2, \widehat{B}_2; \mathbf{x}, t)$ with

$$\widehat{A}_2 = \begin{bmatrix} 0 & c & 1 & 0 & 0 & 0 \\ 1 & 0 & a & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & 1 & c \\ 0 & \lambda_2 & 0 & b & 0 & 1 \\ 0 & 0 & \lambda_3 & 1 & a & 0 \end{bmatrix}, \qquad \widehat{B}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.44}$$

and $\lambda = (\lambda_1, \lambda_2, \lambda_3, 0, 0, 0)^T$. Matrix $\widehat{A}_2$ is of rank 2 if $abc + 1 = 0$ and $\lambda_3 = -ab\lambda_1 + a\lambda_2$. In this case the kernel of $\widehat{A}_2$ is spanned by the two vectors $E = \{(ab, -a, 1, 0, 0, 0), (-a\lambda_2, a\lambda_1, 0, -a, -1/b, 1)\}$. According to Proposition 4.9, we obtain two first integrals

$$I_1 = x_1^{-a\lambda_2} x_2^{a\lambda_1} e^{-ax_1 - x_2/b + x_3}, \tag{4.45.a}$$
$$I_2 = x_1^{ab} x_2^{-a} x_3. \tag{4.45.b}$$

Note that $\log(I_1)$ and $\log(I_2)$ are actually logarithmic first integrals as described in Section 2.2. Interestingly, the existence of such integrals can be related to some analytic properties of the solutions in complex time. Such a relationship will be further explored in Section 5.5. ■

4.8.2 Degeneracy of matrix B

If the matrix $\widehat{B}$ (or, equivalently B itself) is degenerate, then the system can be decoupled.

Definition 4.9 A system $S : \dot{\mathbf{x}} = f(\mathbf{x}) \in \mathcal{Q}_n$ is *decoupled* if, up to a permutation of indices, there exists $r < n$ such that

$$\frac{\partial f_i}{\partial x_j} = 0, \forall\ i = 1, \ldots, r;\ j = r+1, \ldots, n. \tag{4.46}$$

This notion of decoupling is important in dynamical systems. It implies that a decoupled system S can be reduced to two subsystems formally written $S = S_1 \otimes S_2$ with $S_1 \in \mathcal{Q}_r$ evolving in a r-dimensional phase space.

Proposition 4.10 *If matrix B is of rank $r < n$, then there exists a transformation T_C such that the system $T_C(S(A, B))$ decouples into two subsystems $S = S_1 \otimes S_2$, where $S_1 \in \mathcal{Q}_r$, where S_2 is a linear system of ODEs with time-dependent coefficients.*

Proof. If B is of rank $r < n$ then there exists $C \in \mathrm{GL}(n)$ such that $B' = BC = \begin{bmatrix} b & 0 \end{bmatrix}$ and $A' = C^{-1}A = \begin{bmatrix} a^{(1)} \\ a^{(2)} \end{bmatrix}$, where $b \in \mathrm{M}_{m,r}$ is of rank r and $\mathbf{a}_1 \in \mathrm{M}_{r,m}$. The new system, $T_C(S) = S(A', B', \mathbf{x}, t)$, is decoupled and can be written as $S_1 \otimes S_2$ where $S_1(a_1, b; \mathbf{x}, t) \in \mathcal{Q}_r$ and S_2 is a linear system which reduces to quadratures and is given by

$$\dot{x}_i = x_i \sum_{j=1}^{r} a_{2,(i,j)} \prod_{k=1}^{r} x_k^{b_{jk}}, \quad j = r+1, \ldots, n. \tag{4.47}$$

□

4.9 Transformation of first integrals

Another interesting application of this formalism is the transformation of first integrals under quasimonomial transformations. Consider a system $S(f; \mathbf{x}, t) \in \mathcal{Q}_n$ and assume that it admits a rational first integral $I = I(\mathbf{x}, t)$. Now, apply a transformation $T_C(S)$ on the system. If the matrix C is such that $C^{-1} \in \mathrm{GL}(n, \mathbb{Z})$, then there exists a rational first integral for the system $T_C(S)$ obtained by the change of variables

$$I'(\mathbf{x}') = I\left(T_{C^{-1}}(\mathbf{x})\right). \tag{4.48}$$

Lemma 4.1 *If $I(\mathbf{x}')$ is a rational first integral of the canonical form $S_{LV}(M, I; \mathbf{x}', t)$ where $M \in \mathrm{M}_m$, then there exists a rational first integral for the system $S(A, B; \mathbf{x}, t)$, where $M = BA$ and $B \in \mathrm{M}_{m,n}(\mathbb{Z})$.*

The converse proposition requires some further assumptions on matrix B.

Lemma 4.2 *If $I(\mathbf{x})$ is a rational first integral for the system $S(A, B; \mathbf{x}, t)$ ($\{A, B\} \in \{\mathrm{M}_{n,m}, \mathrm{M}_{m,n}(Z)\}$) where B is such that there exists a submatrix $b \in \mathrm{M}_n$ with $\det(b) = \pm 1$, then there exists a rational first integral for the canonical form $S_{LV}(M, I; \mathbf{x}', t)$, where $M = BA$.*

Proof. If there is an $n \times n$ submatrix b of B in the unimodular group, then there exists a matrix $D \in \mathrm{GL}(n, Z)$ such that $BD = \mathrm{I}$. Taking $C = D^{-1}$ in the quasimonomial transformation, we obtain the desired result. □

4.10 An algorithm for polynomial first integrals

The quasimonomial formalism can be used to write an elegant algorithm for the search of polynomial first integrals. The idea is to reduce all computations to simple operations on matrices. We follow the main lines of the algorithm presented in Section 2.4. That is, we look for a first integral of given degree D in the variables $\mathbf{x}$. Let

$$I_D(\mathbf{x}, t) = e^{\chi t} \sum_{i=1}^{p} c_i \prod_{k=1}^{n} x_k^{E_{ik}}, \tag{4.49}$$

where $E \in \mathcal{M}_{n\times p}(\mathbb{N})$ with $|\mathbf{E}_{i.}| \leq D \ \forall \ i$ and $\chi \in \mathbb{C}$. We introduce p new variables corresponding to the monomials appearing in I_D,

$$y_i = \prod_{j=1}^{n} x_j^{E_{ij}}, \qquad i = 1, \dots, p. \tag{4.50}$$

The variable y_i obeys a simple differential equation given by

$$\frac{\dot{y}_i}{y_i} = E_{ij} \frac{\dot{x}_j}{x_j} = \sum_{j=1}^{n} (EA)_{ij} \prod_{k}^{n} x_k^{B_{jk}}. \tag{4.51}$$

A linear combination of the variables yields

$$\sum_{i=1}^{p} c_i \dot{y}_i = \sum_{i=1}^{p} c_i \sum_{j=1}^{n} (EA)_{ij} \prod_{k=1}^{n} x_k^{B_{jk}+E_{ik}}, \qquad i = 1, \dots, p, \tag{4.52}$$

where we have replaced y_i by its explicit values in terms of $\mathbf{x}$. The linear terms in y_i can be singled out by introducing some arbitrary coefficients α_i. That is, we write

$$\begin{aligned} \sum_{i=1}^{p} c_i \dot{y}_i = \sum_{i=1}^{p} c_i y_i \alpha_i & \\ + \sum_{i=1}^{p} c_i y_i (1-\alpha_i) \prod_{k=1}^{n} x_k^{E_{ik}} &+ \sum_{i=1}^{p} c_i \sum_{j=1}^{n} (EA)_{ij} \prod_{k=1}^{n} x_k^{B_{jk}+E_{ik}}, \\ & i = 1, \dots, p. \end{aligned} \tag{4.53}$$

In order for I_D to be a first integral, the following conditions must be satisfied:

$$c_i \alpha_i = c_i \chi, \qquad i = 1, \dots, p, \tag{4.54.a}$$

$$\sum_{i=1}^{p} c_i y_i (1-\alpha_i) \prod_{k=1}^{n} x_k^{E_{ik}} \sum_{i=1}^{p} c_i \sum_{j=1}^{n} (EA)_{ij} \prod_{k=1}^{n} x_k^{B_{jk}+E_{ik}} = 0, \qquad i = 1, \dots, p. \tag{4.54.b}$$

By identifying the coefficients of equal powers, a set of equations for the coefficients c_i is obtained in terms of the parameters of the system. This algorithm was successfully implemented by Codutti (1992).

Example 4.8 The Rikitake system. As an example, consider the Rikitake system (see Example 3.27)

$$\dot{x} = -\mu x + \beta y + yz, \tag{4.55.a}$$
$$\dot{y} = -\mu y + \beta x + xz, \tag{4.55.b}$$
$$\dot{z} = -xy + \alpha, \tag{4.55.c}$$

with α, $\beta \in \mathbb{R}$. To illustrate this method, we look for polynomial first integrals of degree $D = 2$ for the case $\beta = 0$. The corresponding representation of the system $S(A, B, \mathbf{x} = (x, y, z), t)$ is then

$$A = \begin{bmatrix} -\mu & 1 & 0 & 0 & 0 \\ \mu & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}. \tag{4.56}$$

The matrix E of all exponents for polynomials of degree 2 in 3 variables is

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}. \tag{4.57}$$

Let $\mathbf{c} = (c_1, \dots, c_9)$ be the corresponding coefficients of I_2. After substitution, we obtain three sets of equations

$$\alpha D_1 = 0, \tag{4.58.a}$$
$$2\alpha D_2 = 0, \tag{4.58.b}$$
$$D_4 = 0, \tag{4.58.c}$$
$$D_7 = 0, \tag{4.58.d}$$
$$D_8 = 0, \tag{4.58.e}$$

$$D_3 - \mu D_7 = D_7 \chi \tag{4.59.a}$$
$$\alpha D_4 - \mu D_3 = D_3 \chi, \tag{4.59.b}$$
$$D_6 - \mu D_4 = D_4 \chi \tag{4.59.c}$$
$$\alpha D_7 - \mu D_6 = D_6 \chi, \tag{4.59.d}$$

$$-D_1 - 2\mu D_8 = D_8 \chi, \tag{4.60.a}$$
$$-2D_2 + 2D_5 + 2D_9 = 0. \tag{4.60.b}$$

A simple solution of this system is given by $\mathbf{c} = (0, 0, 0, 0, -1, 0, 0, 0, 1)$ and $\chi = -2\mu$. The corresponding first integral is

$$I_2 = (x^2 - y^2)e^{-2\mu t}. \tag{4.61}$$

■

4.11 Jacobi's last multiplier for quasimonomial systems

In order to apply Jacobi's last multiplier theorem (see Section 2.11), a multiplier associated with the invariant measure for the vector field $\delta_{\mathbf{f}}$ has to determined. That is, we look for a function $G = G(\mathbf{x})$ such that

$$\partial_{\mathbf{x}}(G\mathbf{f}) = 0. \tag{4.62}$$

A simple choice for the function G for quasimonomial systems is $G = \mathbf{x}^{\boldsymbol{\alpha}}$.

Proposition 4.11 *Let $S(A, B; \mathbf{x}, t)$ be a quasimonomial system and $M = BA$ and define $\mathbf{v} \in \mathbb{C}^n$ to be the vector of components $v_i = -M_{ii} - \sum_{j=1}^{m} M_{ji}$. If $\boldsymbol{\eta}.\mathbf{v} = 0$, $\forall\, \boldsymbol{\eta} \in \mathrm{Ker}(M)$, the system $S(A, B)$ admits a Jacobi multiplier of the form $G = \mathbf{x}^{\boldsymbol{\alpha}}$ where $\boldsymbol{\alpha} = \boldsymbol{\beta} B$ and $\boldsymbol{\beta}$ is a solution of $M^T\boldsymbol{\beta} = \mathbf{v}$.*

Proof. Consider the canonical form $S(M, \mathrm{I}; \mathbf{y}, t)$ associated with the system $S(A, B; \mathbf{x}, t)$. If $S(M, \mathrm{I})$ admits a multiplier of the form $G = \mathbf{y}^{\boldsymbol{\beta}}$, then the original system $S(A, B)$ admits, via a quasimonomial transformation, the multiplier $G = \mathbf{x}^{\boldsymbol{\alpha}} = \mathbf{x}^{\boldsymbol{\beta} B}$. Now consider the Lotka-Volterra canonical form $S(M, \mathrm{I}; \mathbf{y}, t)$ and apply condition (4.62) to obtain

$$\sum_{i=1}^{m} \frac{\partial}{\partial y_i}(\mathbf{y}^{\boldsymbol{\beta}} y_i M_{ij} y_j) = \mathbf{y}^{\boldsymbol{\beta}} \sum_{i=1}^{m} \left((\beta_i + 1)M_{ij} + M_{ii}\right) y_j. \tag{4.63}$$

This relation must hold for all y_j. Hence, we have

$$\sum_{i=1}^{m} \beta_i M_{ij} = -M_{jj} - \sum_{i=1}^{m}(M_{ij}). \tag{4.64}$$

That is, $M^T\boldsymbol{\beta} = \mathbf{v}$. This linear system has a solution if and only if

$$\boldsymbol{\eta}.\mathbf{v} = 0 \;\forall\, \boldsymbol{\eta} \in \mathrm{Ker}(M). \tag{4.65}$$

□

In particular if $\det(M) \neq 0$, the condition for the existence of a multiplier is trivially satisfied and we find that $G = \mathbf{x}^{\boldsymbol{\alpha}}$ is a Jacobi multiplier with $\boldsymbol{\alpha} = (M^T)^{-1}\mathbf{v}$ (see Maciejewski (1998) for a similar result and Example 2.26 for an application).

Example 4.9 The May-Leonard system (1975). Consider the Lotka-Volterra system $S(M, \mathrm{I}; \mathbf{x}, t)$ in $\mathbb{R}^3$ with

$$M = \begin{bmatrix} 1 & a & b \\ b & 1 & a \\ a & b & 1 \end{bmatrix}. \tag{4.66}$$

According to Proposition 4.11, the vector $\mathbf{v}$ is $\mathbf{v} = -(a + b + 2)(1, 1, 1)$. There are two cases of interest. First, if $\det(M) \neq 0$ then, the system admits a Jacobi multiplier $G = \mathbf{x}^{\boldsymbol{\beta}}$ where $\boldsymbol{\beta}$ is given by the solution of (4.64), that is, $\boldsymbol{\beta} = -\frac{2+b+a}{a+b+1}(1, 1, 1)$. Note that in the particular case when the system is volume preserving, that is, $a + b + 2 = 0$, and $\boldsymbol{\beta} = (0, 0, 0)$, there exists a first integral (Strelcyn & Wojciechowski, 1988) given by

$$I = \frac{(x_1 + x_2 + x_3)^3}{x_1 x_2 x_3}. \tag{4.67}$$

The Jacobi last multiplier theorem provides the last first integral and the system is completely integrable.

Second, if $\det(M) = 0$, then either $a + b + 1 = 0$ or $a^2 - a - ab - b + 1 + b^2 = 0$. In both of these cases there exists a quasimonomial first integral $I = \mathbf{x}^m$ where $\mathbf{m} \in \mathrm{Ker}(M^T)$. Now, in the first case $(a + b + 1 = 0)$, $\mathbf{m} = \boldsymbol{\eta} = (1, 1, 1) \in \mathrm{Ker}(M)$, $\boldsymbol{\eta}.\mathbf{v} \neq 0$, and there is no quasimonomial Jacobi multiplier. In the second case, $\boldsymbol{\eta}.\mathbf{v} = 0$ and there exists a Jacobi multiplier $\boldsymbol{\beta} = -\frac{2+b+a}{a+b+1}(1, 1, 1)$ where (a, b) are constrained by $a^2 - a - ab - b + 1 + b^2 = 0$ and we conclude that the system is completely integrable. ■

4.12 Application: semi-simple normal forms

Truncated normal forms are polynomial vector fields. The theory of such vector fields is a basic tool for the local analysis of dynamical systems in phase space, that is, the analysis of the solutions in a neighborhood of their fixed points (Bibikov, 1979; Chow & Hale, 1982; Elphick *et al.*, 1987). Here, we study semi-simple normal forms, that is, normal forms with linear diagonal parts as a particular class of vector fields, and we show that general properties of their solutions can be obtained.

Definition 4.10 The class $\mathcal{N}_n \in \mathcal{P}_n$ of *semi-simple normal forms* is the set of systems $\widehat{S}(\widehat{A}, \widehat{B}, \boldsymbol{\lambda})$ with a linear diagonal part diag$(\boldsymbol{\lambda})$ as defined in (4.37) with $|B_{i.}| > 1 \ \forall \ i$ and $\widehat{B}\boldsymbol{\lambda} = \mathbf{0}$.

The relation $\widehat{B}\boldsymbol{\lambda} = \mathbf{0}$ is the *resonance condition* introduced in Section 3.9.4. For a given set of ODEs with linear eigenvalues $\boldsymbol{\lambda}$, the monomial $q = \prod_{i=1}^{n} x_i^{\mathbf{b}_i}$ is *resonant* for the ith equation if $\lambda_i - \mathbf{b}.\boldsymbol{\lambda} = \mathbf{0}$.

The resonance condition provides the normal forms with a very peculiar structure from which general statements can be deduced (Walcher, 1991). For instance, the following statement can be easily proved.

Proposition 4.12 *Every semi-simple normal form can be decoupled by a quasimonomial transformation.*

Proof. Consider a quasimonomial system with a semi-simple linear part around the origin. Assume without loss of generality that the linear part is diagonal with linear eigenvalues $\boldsymbol{\lambda}$. That is, the system can be written $\widehat{S}(\widehat{A}, \widehat{B}, \boldsymbol{\lambda})$. Now, the resonance condition $\widehat{B}\boldsymbol{\lambda} = 0$ implies that matrix B is degenerate of rank $r < n$. Therefore, Proposition 4.10 can be applied, and there exists a transformation T_C built on ker(B) which decouples the normal form. □

Geometrically, the resonance condition states that the Newton polyhedron built on the matrix B is at most of dimension $(n-1)$. In particular, in two dimensions, the Newton polygon reduces to the line $\widehat{B}\boldsymbol{\lambda} = 0$ which can be mapped by a quasimonomial transformation T_C on one of the axes, which has the effect of decoupling the system.

The two following results are a direct consequence of the last proposition. First, every two-dimensional semi-simple normal form can be decoupled and an exact closed form solution can be obtained (see also Bruno (1999)). Second, semi-simple normal forms in three dimensions are equivalent to a two-dimensional flow and as a consequence do not exhibit chaotic motion (this is a consequence of the classical result of Poincaré which states that autonomous two-dimensional vector fields on the plane do not exhibit chaotic motion (Guckenheimer & Holmes, 1983)). This last result does not hold in higher dimensions since for $n \geq 4$, chaotic normal forms are known to exist (Hoveijn & Verhulst, 1990). The use of the quasimonomial formalism in normal form theory has been further investigated by Brenig and Goriely (1994) and Louies and Brenig (1999).

4.13 Quasimonomial transformation and the Painlevé property

The main difficulty encountered when we try to relate integrability properties to the Painlevé test is associated with coordinate transformations. In general, if a system satisfies the Painlevé test in a given set of variables, this property will not hold in another set of variables. For instance, consider the simple Bernoulli equation

$$\dot{x} + f(t)x^n + g(t) = 0. \tag{4.68}$$

This system does not have the Painlevé property for $n > 2$ in the set of variables $\{x, t\}$ since the solution has a movable critical branch point given by $x \sim (t - t_*)^{\frac{1}{n-1}}$. However, it is well-known that the simple change of variables $x = y^{\frac{1}{n-1}}$ maps the Bernoulli equation to a linear equation with no movable singularities. The transformation contains the information on the branching, and the solution is single-valued in the set of variables $\{y, t\}$.

Example 4.10 A logarithmic first integral. A less trivial example is provided by the system

$$\dot{x}_1 = \lambda_1 x_1 + m_1 x_1 x_2, \tag{4.69.a}$$

$$\dot{x}_2 = \lambda_2 x_2 + m_2 x_1 x_2. \tag{4.69.b}$$

This system does not pass Painlevé test #2 as detailed in Chapter 3 due to the existence of a logarithmic branch point. Is there a change of variables such that in the new variables the system exhibits the Painlevé property? Note that this system admits the logarithmic first integral

$$I(x_1, x_2) = (\lambda_2 x_1 - \lambda_1 x_2)(m_1 \log x_1 + m_2 \log x_2). \tag{4.70}$$

In Chapter 5, it will be shown that the existence of a logarithmic integral implies that the system does not satisfy the Painlevé property. However, the question remains: is there a way to obtain such first integrals using singularity analysis and defining a suitable Painlevé test? ■

If we want to obtain an integrability criterion based on singularity analysis, it is clear that it should be invariant under coordinate transformations. Hence, we have to take into account not only systems satisfying the Painlevé test, but also any system that can be mapped onto a system having such a property. More precisely, the main idea of this section is to define a Painlevé test for an equivalence class. A system will pass *the Painlevé test with respect to an equivalence class* if there exists at least one member of the class which fulfills the condition of the Painlevé test #2. The problem lies now in the choice of transformations and in the computation of the members of the equivalence class satisfying the Painlevé test as well as in a rigorous classification of the different equivalence classes. The Painlevé test can naturally be extended to the equivalence classes discussed in the previous sections (with respect to the quasimonomial transformation and the new-time transformations). In doing so, we can define a large class of systems whose integrability is related to analytic properties in the complex plane of time. This approach has also the advantage of linking explicitly critical singularities with the better known full Painlevé property (Adler & van Moerbeke, 1989b; Ercolani & Siggia, 1989). The specific connection between the Painlevé property and different notions of integrability will be discussed further in Chapter 5.

4.13.1 The transformations group of the Riemann sphere

The first group of transformations to be considered are those of one dependent and one independent variable leaving the singularities unchanged. These transformations will not alter the Painlevé property. An *automorphism* $U : \Sigma \to \Sigma$ of the Riemann sphere into itself is a meromorphic bijection, the set of all automorphisms of Σ is denoted by $\text{Aut}(\Sigma)$.

Proposition 4.13 *(Jones & Singerman, 1987) Aut(Σ) consists of the functions*

$$U(a, b, c, d; x) = \frac{ax + b}{cx + d}, \tag{4.71}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Transformations of this type are also known as *homographic, linear fractional* or *Möbius* transformation. The transformation U is determined uniquely by the parameters, up to a scaling $(a, b, c, d) \to (\lambda a, \lambda b, \lambda c, \lambda d)$. An important property of these transformation is that $\text{Aut}(\Sigma)$ is a group of homeomorphisms from the Riemann sphere to itself. This group is commonly called the Möbius group. It is easy to see using a matrix notation for the parameters, and taking into account the scaling of the parameters that $\text{Aut}(\Sigma)$ coincides with the projective special linear group $\text{PSL}(2, \mathbb{C})$, the image of $\text{SL}(2, \mathbb{C})$ in the quotient-group $\text{PGL}(2, \mathbb{C})/\text{K}$ where $\text{K}=\{\text{diag}(\lambda, \lambda); \lambda \in \mathbb{C}_0\}$.

The interest in these transformations is that they preserve the nature of the solutions of nth order equations in one variable. It is the first example of transformation of equations and the one used for the Painlevé classification of second-order ODEs. The homographic transformations naturally define equivalence classes among nth order differential equations which have been used for classification purposes. Consider the nth order equations

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 = 0, \tag{4.72}$$

where the functions a_i are analytic (in t).

Theorem 4.1 *Consider the homographic transformation, $H : \Sigma \times \mathbb{C} \to \Sigma \times \mathbb{C}$*

$$H : (x, t) \to (X, T) = (U(A(t), B(t), C(t), D(t); x), T(t)), \tag{4.73}$$

where $U \in \mathrm{Aut}(\Sigma)$ and $T(t), A(t), B(t), C(t)$, and $D(t)$ are analytic functions of t. If equation (4.72) has the Painlevé property, then the equation obtained by the homographic transformation, H, also enjoys the Painlevé property.

Moreover, it can be shown that these transformations are the only possible transformations leaving the Painlevé property for nth order equations invariant. This is due to the fact that $\mathrm{Aut}(\Sigma)$ contains all possible bijections of the Riemann sphere.

4.14 Painlevé tests and quasimonomial transformations

The quasimonomial class can be divided into equivalence classes by the associated quasimonomial transformation. In each class, there exists a canonical form. The other members of the equivalence class can be formally obtained by applying a quasimonomial transformation T_C. It was noted previously that this equivalence is formal. Now, we use the canonical form S_{LV} as a unique representative of the whole equivalence class for the Painlevé test. First, we analyze the dominant balances $\mathcal{F}$ of a given system $S(A, B) \in \mathcal{Q}_n$ in terms of its matrices $\{A, B\}$. Second, we give a geometric picture of the dominance relation in terms of the Newton polyhedron while emphasizing its natural link with normal forms analysis. Third, we study the transformation of Laurent series solutions of $S(A, B)$ under the quasimonomial transformation to show that the invariant matrix M contains all the information required for the Painlevé test.

4.14.1 The dominant balances

Consider a system $S(A, B; \mathbf{x}, t) \in \mathcal{Q}_n$ with $\{A, B\} \in \{\mathrm{M}_{n,m}, \mathrm{M}_{m,n}\}$. We are interested in finding a dominant balance $\mathcal{F}$. The first step is to find a weight-homogeneous decomposition of the vector field $S(A, B) = S(\mathbf{f})$ in terms of the matrices A and B. According to Section 3.8, the first step of the singularity analysis consists of finding a decomposition of the vector field $\mathbf{f}$ of the form

$$\mathbf{f} = \mathbf{f}^{(0)} + \mathbf{f}^{(1)} + \ldots + \mathbf{f}^{(m')}, \tag{4.74}$$

so that $\mathbf{x}^{(0)} = \boldsymbol{\alpha}(t - t_*)^{\mathbf{p}}$ is an exact solution of $\dot{\mathbf{x}} = \mathbf{f}^{(0)}$ and

$$\mathbf{f}^{(i)}(\boldsymbol{\alpha}(t - t_*)^{\mathbf{p}}) = \boldsymbol{\gamma}^{(i)}(t - t_*)^{\mathbf{p}+q^{(i)}-1}, \tag{4.75}$$

with $q^{(i)} > q^{(j)} > 0$ for all i, j such that $i > j > 0$. The vector $\mathbf{p}$ is the vector of *dominant exponents* and the numbers $q^{(i)}$ are the *non-dominant exponents*. A consistent decomposition of $\mathbf{f} = \mathbf{x}A\mathbf{x}^B$ is provided by the decomposition of matrices $\{A, B\}$

$$A = \begin{bmatrix} A^{(0)} \ A^{(1)} \ldots A^{(m')} \end{bmatrix}, \quad B = \begin{bmatrix} B^{(0)} \\ B^{(1)} \\ \vdots \\ B^{(m')} \end{bmatrix}, \tag{4.76}$$

with $m' < m - n$, $\{A^{(i)}, B^{(i)}\} \in \{\mathrm{M}_{n,l_i}, \mathrm{M}_{l_i,n}\}$, and $\sum_{i=1}^{m'} l_i = m$. This decomposition is defined up to a simultaneous permutation of the rows of B and columns of A such that the dominant part of the vector field is always represented by the first rows (resp. columns) of matrix B (resp. A). In order for this decomposition to define a dominant balance $\mathcal{F}$, we must check two conditions. First, the system $S(A^{(0)}, B^{(0)})$ must support a scale-invariant solution $\mathbf{x}^{(0)} = \boldsymbol{\alpha}(t-t_*)^{\mathbf{p}}$. That is, a non-vanishing solution $\boldsymbol{\alpha}$ must exist for the system

$$\boldsymbol{\alpha} A^{(0)} \boldsymbol{\alpha}^{B^{(0)}} = \boldsymbol{\alpha}\mathbf{p}, \tag{4.77.a}$$
$$B^{(0)}\mathbf{p} = -\mathbf{1}. \tag{4.77.b}$$

Second, we must check that the higher-order terms of the decomposition are defined with respect to $\mathbf{p}$, that is,

$$q^{(i)} = B^{(i)}\mathbf{p} + \mathbf{1}, \qquad i = 1, \ldots, m' \quad (m' < m - n), \tag{4.78}$$

with $q^{(i)} < q^{(j)}\ \forall\ i < j$ and $q^{(0)} = 0$.

To summarize, the first step of the singularity analysis can be split into two conditions. First, we find a decomposition of matrix B such that

$$\mathbf{b}_i.\mathbf{p} = \mathbf{b}_j.\mathbf{p}\ \forall\ \mathbf{b}_i, \mathbf{b}_j \in B^{(0)}, \tag{4.79.a}$$
$$\mathbf{b}_i, \mathbf{p} > \mathbf{b}_j.\mathbf{p}\ \forall\ \mathbf{b}_i \in B^{(k)},\ \mathbf{b}_j \in B^{(l)} \text{ with } k > l. \tag{4.79.b}$$

Second, for the matrices $A^{(0)}$ and $B^{(0)}$ defined by the relations (4.79), we show that there exists a non-vanishing solution $\boldsymbol{\alpha} \in \mathbb{C}^n$ for system (4.77.a). The *order* of a balance is defined by the number of vanishing coefficients for the vector $\boldsymbol{\alpha}$ and is an important notion in the classification of different dominant balances.

A particularly instructive decomposition of system S is obtained by the choice of $\{A^{(0)}, B^{(0)}\} \in \{\mathrm{GL}(n,\mathbb{C}), \mathrm{GL}(n,\mathbb{K})\}$. Then, the exponents of the scale-invariant solution $\mathbf{x} = \boldsymbol{\alpha}(t-t_*)^{\mathbf{p}}$ are given by $B^{(0)}\mathbf{p} = -\mathbf{1}$, that is,

$$\mathbf{p} = (B^{(0)})^{-1}(-\mathbf{1}). \tag{4.80}$$

In the same way, one possible choice of $\boldsymbol{\alpha}$ is given by the solutions of the system

$$\boldsymbol{\alpha}^{B^{(0)}} = \left(M^{(0)}\right)^{-1}(-\mathbf{1}), \tag{4.81}$$

with $M^{(0)} = B^{(0)}A^{(0)}$.

Note that $\boldsymbol{\alpha}$ is uniquely determined by the last equation if and only if B is in the unimodular group defined by the conditions $\det(B^{(0)})=\pm 1$ and $\mathbb{K} = \mathbb{Z}$). In general, there are different balances and each solution of the system uniquely defines a different balance $\mathcal{F}$ with no arbitrary constant. We can check that the vector field $S(A^{(0)}, B^{(0)})$ is dominant near a singularity by showing that the non-dominant part characterized by $\{A^{(i)}, B^{(i)}\}$ can be neglected as $(t-t_*) \to 0$. To do so, we compute the $q^{(i)}$'s from relation (3.55). According to the different values of the weights $q^{(i)}$, we introduce the decomposition of A and B by the condition

$$q^{(i)} = B^{(i)}\mathbf{p} + 1, \qquad i = 1, \ldots, m' \quad (m' < m - n), \tag{4.82}$$

with $q^{(i)} < q^{(j)}\ \forall\ i < j$.

Example 4.11 As a first example, consider the planar system

$$\dot{x}_1 = \lambda_1 x_1 + x_1^2 x_2^2 - 2\mu x_1^4 x_2^3, \tag{4.83.a}$$
$$\dot{x}_2 = \lambda_2 x_2 + x_1^2 + \mu x_1^3 x_2^4. \tag{4.83.b}$$

In terms of matrices A and B, this system reads

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & -2\mu \\ \lambda_2 & 0 & 1 & \mu \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 2 & -1 \\ 3 & 3 \end{bmatrix}. \tag{4.84}$$

The dominant behaviors can be determined according to relations (4.79.a). First, we consider the balances of order 2. We must consider the cases corresponding to two different decompositions.

- **Case 1.** The first possible decomposition of order 2 is given by the matrices

$$B^{(0)} = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}, \quad B^{(1)} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} 2 & -1 \end{bmatrix}. \tag{4.85}$$

The corresponding decomposition of matrix A is

$$A^{(0)} = \begin{bmatrix} 1 & -2\mu \\ 0 & \mu \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.86}$$

According to this decomposition, we find

$$\mathbf{p} = (1/3, -2/3), \quad q^{(0)} = 0, \quad q^{(1)} = 1, \quad q^{(2)} = 7/3. \tag{4.87}$$

It is now possible to compute the leading coefficients $\boldsymbol{\alpha}$ from (4.77.a). Six different leading terms are found for this unique value of $\mathbf{p}$ giving rise to six different balances of order 2:

$$\boldsymbol{\alpha} = (-3, 1)\,(27\mu)^{-1/6}\, e^{ik\pi/3}, \qquad k = 0, \dots, 5. \tag{4.88}$$

- **Case 2.** Another decomposition of the vector field is given by the choices

$$B^{(0)} = \begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}, \quad B^{(1)} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \tag{4.89}$$

and

$$A^{(0)} = \begin{bmatrix} 0 & -2\mu \\ 1 & \mu \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}. \tag{4.90}$$

The new values of $\mathbf{p}$ and $\boldsymbol{\alpha}$ are

$$\mathbf{p} = (-4/9, 1/9), \quad q^{(0)} = 0, \quad q^{(1)} = 7/9, \quad q^{(2)} = 1, \tag{4.91}$$

$$\boldsymbol{\alpha} = (-3/2, 1)\left(\frac{27\mu}{4}\right)^{-1/6} e^{ik\pi/3} \qquad k = 0, \dots, 5. \tag{4.92}$$

For this system, there is no dominant balance of order 1 and we conclude that it supports 12 different balances of order 2, that is, 12 different types of local expansion around its movable singularities. ■

4.14.2 Dominant balance and Newton's polyhedron

Now, we give a geometric interpretation of the decomposition of vector fields for a dominant balance in terms of the Newton polyhedron. In Section 4.5, we saw that for each vector field $S(A, B; \mathbf{x}, t)$ we can build a Newton polygon associated with matrix B. Finding a dominant balance $\mathcal{F}$ reduces to finding a proper decomposition of matrix B. Therefore, we have to find $B^{(0)}$ and $\mathbf{g} \in \mathbb{K}^n$ such that

$$\mathbf{b}_i.\mathbf{g} = \mathbf{b}_j.\mathbf{g} \quad \forall\ \mathbf{b}_i,\ \mathbf{b}_j \in B^{(0)}, \tag{4.93.a}$$

$$\mathbf{b}_k.\mathbf{g} < \mathbf{b}_j.\mathbf{g} \quad \forall\ \mathbf{b}_k \in B \setminus B^{(0)},\ \mathbf{b}_j \in B^{(0)}, \tag{4.93.b}$$

where, following Section 4.5, B is a set of m points in $\mathbb{K}^n$. Relation (4.93) is precisely the one introduced in Section 4.5 to define a *boundary subset* $B^{(0)}$ of B. To identify the subset $B^{(0)}$ with a dominant balance, we require that $\mathbf{b}_i.\mathbf{g} = 1$ for all $\mathbf{b}_i \in B^{(0)}$. Therefore, the dominant exponents are given by $\mathbf{p} = -\mathbf{g}$ since relation (4.93) is equivalent to condition (4.79.a). Similarly, relation (4.93) was used in Section 4.5 to define the Newton polyhedron (see Equation (4.25)).

Proposition 4.14 *Each dominant balance $\mathcal{F}$ for $S(A, B; \mathbf{x}, t)$ of order l defines a subset $B^{(0)}$ which is an l-dimensional face $\Gamma^{(l)}$ of the Newton polyhedron of $S(A, B)$.*

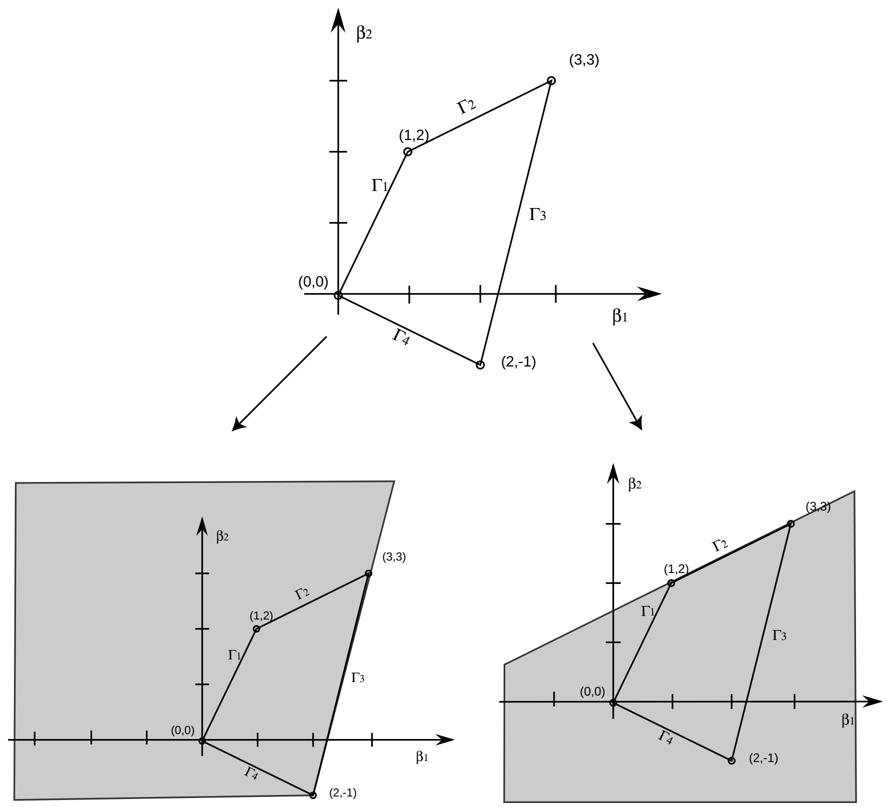

Figure 4.5: The Newton polygon of the set B together with the two dominance conditions for the first (left) and second decomposition (right) (Case 1 and 2). The dominance condition is indicated by the shaded area.

Example 4.12 Continuation of the previous example. Consider again the planar system (4.83). Matrix B defines a Newton polygon with four faces $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ (see Figure 4.5). Although each face corresponds to a possible balance, only two of them are dominant, namely, Γ_2 and Γ_3. In Figure 4.5, the dominance condition (4.93) for the decomposition (4.85) is represented by the shaded area. The dominance condition is fulfilled since the points defined by sets $B^{(1)}$ and $B^{(2)}$ are in the (supporting) half-plane defined by relation (4.93). The dominance condition for the second decomposition is also represented in Figure 4.5. The vector $\mathbf{p}$ for $\mathcal{F}$ can be found geometrically by considering the intersection of the line, defined by the face Γ, with the axes. For instance, consider the face Γ_2 (see Figure 4.5) defined by

$$\Gamma_2 \cap \beta_1 = (-1/p_1, 0) = (3, 0), \tag{4.94.a}$$

$$\Gamma_2 \cap \beta_2 = (0, -1/p_2) = (0, 3/2). \tag{4.94.b}$$

That is, define

$$\Gamma_2 : \beta_1 p_1 + \beta_2 p_2 = -1. \tag{4.95}$$

In other words, the condition that $q^{(i)} > 0$ for all i can be represented by the half-plane (the shaded area in Figure 4.5)

$$\beta_1 p_1 + \beta_2 p_2 > -1. \tag{4.96}$$

■

4.14.3 The Kovalevskaya exponents

Consider a dominant balance $\mathcal{F}$ of order l. The Kovalevskaya exponents only depend on the leading part of the vector field $S(A^{(0)}, B^{(0)})$. The Kovalevskaya matrix K, defined by (3.73), is easily obtained from the matrices $A^{(0)}$, $B^{(0)}$,

$$K = A^{(0)}\mathrm{diag}(\boldsymbol{\alpha}^{B^{(0)}})B^{(0)}. \tag{4.97}$$

The Kovalevskaya matrix does not depend on the particular balance $\boldsymbol{\alpha}$ but only on $\boldsymbol{\alpha}^{B^{(0)}}$.

Lemma 4.3 *Let $\mathcal{F}$, $\mathcal{F}'$ be two dominant balances for the vector field $S(A^{(0)}, B^{(0)})$ such that $\boldsymbol{\alpha}^{B^{(0)}} = \boldsymbol{\alpha}'^{B^{(0)}}$, and let K and K' be their Kovalevskaya matrices. Then, $K = K'$.*

We conclude that the two first steps of the singularity analysis can be reduced to simple algebraic computation. Once the parameters $\boldsymbol{\alpha}$ are known, we can compute K, the eigenvalues of which are the Kovalevskaya exponents ρ. Let $\mathcal{R}$ be the set of eigenvalues of K and $\mathcal{R}^{(+)}$ the set of positive eigenvalues of K. We show in the next section that the third step of the singularity analysis, that is, checking the compatibility conditions , can also be reduced to a simple algebraic problem and that a simpler form of K can be obtained owing to the invariance of the Kovalevskaya exponents with respect to the QMTs.

Example 4.13 The same example. Consider the 12 different dominant balances for System (4.83). The Kovalevskaya exponents can be computed using (4.97). According to Lemma 4.3, the Kovalevskaya exponents do not depend on $\boldsymbol{\alpha}$ but rather on $\boldsymbol{\alpha}^{B^{(0)}}$. Therefore, we have

$$\mathcal{R}^{(1)} = \{-1, 2\}, \qquad \mathcal{R}^{(2)} = \{-1, \frac{4}{9}\}. \tag{4.98}$$

■

4.14.4 Quasimonomial transformation of local series

A complete description of the dominant balances and of the Kovalevskaya exponents for the system $S(A, B; \mathbf{x}, t) \in \mathcal{Q}_n$ has been obtained in terms of the matrices A and B. We want to extend this analysis to the whole equivalence class obtained by applying quasimonomial transformations. That is, given the dominant balances $\mathcal{F}$ and the corresponding Kovalevskaya exponent matrices K of a system S, what can we say about the dominant balance $\mathcal{F}'$ and the matrices K' for the system $T_C(S)$? We want to establish a correspondence based solely on the transformation matrix C. In the following, $(\ )'$ denotes the corresponding elements in the new set of variables $\{x', t\}$.

Consider a dominant balance $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$. We know from Lemma 4.14 that a dominant balance is a decomposition of matrix B satisfying relations (4.79.a) and (4.77.a) and that it can be represented by one of the faces of the Newton polygon constructed on matrix B. We also know that the Newton polygon is mapped under a QMT to another Newton polygon (see Proposition 4.7). Therefore, a dominant balance remains dominant under a QMT. The values of $\mathbf{p}'$ and $\boldsymbol{\alpha}'$ can be readily computed using the transformation rules for matrices $\{A, B\}$. That is,

$$\mathbf{p}' = C^{-1}\mathbf{p}, \tag{4.99.a}$$
$$\boldsymbol{\alpha}'^{C} = \boldsymbol{\alpha}, \tag{4.99.b}$$
$$q'^{(i)} = q^{(i)}, \quad i = 1, \ldots, m. \tag{4.99.c}$$

Next, we compute the new Kovalevskaya matrix K' using relation (4.97) and the transformation rule for matrices $\{A, B\}$ under a quasimonomial transformation. We have

$$K' = C^{-1}KC, \tag{4.100}$$

and conclude that the Kovalevskaya exponents are invariant under a QMT.

Proposition 4.15 *Consider a system $S : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \in \mathcal{Q}_n$ and assume it admits the formal local series*

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{\mathbf{i},\ |\mathbf{i}|>1}^{\infty} \mathbf{c}_{\mathbf{i}} \tau^{(\boldsymbol{\rho},\mathbf{i})} \right), \qquad (\boldsymbol{\rho},\mathbf{i}) = \sum_{j=1}^{n+1} \rho_j i_j, \tag{4.101}$$

where $\mathbf{c}_{\mathbf{i}}$ is polynomial in $\log\tau$ and ρ_{n+1} is the least common denominator of $\{q^{(1)}, \ldots, q^{(m)}\}$. Then, the system $S' = T_C(S)$ has the formal local series

$$\mathbf{x} = \tau^{\mathbf{p}'} \left(\boldsymbol{\alpha}' + \sum_{\mathbf{i},\ |\mathbf{i}|>1}^{\infty} \mathbf{c}'_{\mathbf{i}} \tau^{(\boldsymbol{\rho},\mathbf{i})} \right), \qquad (\boldsymbol{\rho},\mathbf{i}) = \sum_{j=1}^{n+1} \rho_j i_j, \tag{4.102}$$

where $\mathbf{p}'$ and $\boldsymbol{\alpha}'$ are given by (4.99).

In particular, when the solutions can be expanded in Puiseux series, we have the following result.

Proposition 4.16 *Consider a system $S : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \in \mathcal{Q}_n$ and assume it admits the local Puiseux solution*

$$\mathbf{x} = \boldsymbol{\alpha}(t - t_*)^{\mathbf{p}} \sum_{i=0}^{\infty} \mathbf{a}_i (t - t_*)^{i/s}. \tag{4.103}$$

Then, the system $S' = T_C(S)$ has also a Puiseux solution given by

$$\mathbf{x} = \boldsymbol{\alpha}'(t - t_*)^{\mathbf{p}'} \sum_{i=0}^{\infty} \mathbf{a}'_i (t - t_*)^{i/s}, \tag{4.104}$$

where $\{\boldsymbol{\alpha}', p'\}$ is given by (4.99).

We can establish necessary conditions for the Painlevé test within the quasimonomial class. From the point of view of integrability, it is sufficient to find one element of an equivalence class that satisfies the Painlevé test. Since the Kovalevskaya exponents are invariant under quasimonomial transformations, we conclude that it is necessary for the non-dominant exponents $q^{(i)}$ and the Kovalevskaya exponents $\boldsymbol{\rho}$ to be integers. Then, there is a quasimonomial transformation mapping the dominant exponents $\mathbf{p}$ onto $\mathbf{p}' \in \mathbb{Z}^n$. This condition, originally formulated in the Painlevé test can therefore be eliminated within this setting.

This analysis also shows that some algebraic branch points cannot be removed by a change of dependent variables. Therefore, the weak-Painlevé test cannot be reduced to a regular Painlevé test within the equivalence class of the quasimonomial systems. Consequently, we will use another set of transformations, the new-time transformations. Before doing so, we complete our analysis by carrying out the third step of the singularity analysis on the canonical representative of the quasimonomial systems, the Lotka-Volterra canonical form.

Example 4.14 **Reaction-diffusion equations.** Consider the reaction-diffusion equation (Powell & Tabor, 1992)

$$U_y = U_{zz} + P(U), \tag{4.105}$$

where $U = U(y, z)$ and $P(U) = \mu U + \lambda U^N - U^{2N-1}$. The *traveling-wave reduction* $t = z - cy$ maps this equation to the ordinary differential equation of second order

$$\ddot{u} + c\dot{u} + P(u) = 0, \tag{4.106}$$

which can be written

$$\dot{x}_1 = x_2, \tag{4.107.a}$$

$$\dot{x}_2 = -cx_2 - \mu x_1 - \lambda x_1^N + x_1^{2N-1}, \tag{4.107.b}$$

where $x_1 = u, x_2 = \dot{u}$. In terms of matrices A and B, system $S(A, B; \mathbf{x}, t)$ reads

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -c & -\mu & -\lambda & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 1 & -1 \\ N & -1 \\ 2N-1 & -1 \end{bmatrix}. \tag{4.108}$$

The first step in the singularity analysis is the decomposition of the vector field. There are $(2N-2)$ balances of order 2 associated with the decomposition

$$\begin{aligned} B^{(0)} &= \begin{bmatrix} -1 & 1 \\ 2N-1 & -1 \end{bmatrix}, & B^{(1)} &= \begin{bmatrix} 0 & 0 \\ N & -1 \end{bmatrix}, & B^{(2)} &= \begin{bmatrix} 1 & -1 \end{bmatrix}, \\ A^{(0)} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & A^{(1)} &= \begin{bmatrix} 0 & 0 \\ -\lambda & -\mu \end{bmatrix}, & A^{(2)} &= \begin{bmatrix} 0 \\ -c \end{bmatrix}, \end{aligned} \tag{4.109}$$

with dominant exponents $\mathbf{p} = (1/(1-N), N/(1-N))$ and non-dominant exponents $q^{(1)} = 1,\ q^{(2)} = 2$. The leading coefficients $\boldsymbol{\alpha} = (\alpha_1, \alpha_2 = p_1\alpha_1)$ are given by the $(2N-2)$ roots of $\alpha_1^{2N-2} = N/(N-1)^2$. This decomposition corresponds to the choice of $\ddot{u} = u^{2N-1}$ as a dominant vector field. The Kovalevskaya exponents can be easily computed $\mathcal{R} = \{-1, 2N/(N-1)\}$. The case $N = 3,\ \lambda = 1$ is of particular interest (van Saarloos, 1988; van Saarloos, 1989). In that case, the Kovalevskaya exponents are integers ($\mathcal{R} = \{-1, 3\}$) but the leading exponents are rational. There are four different balances characterized by different values of α_1: $\mathcal{F}_{1,2}\{\pm\sqrt[4]{\frac{3}{4}}, \mathbf{p}\}$ and $\mathcal{F}_{3,4} = \{\pm i\sqrt[4]{\frac{3}{4}}, \mathbf{p}\}$. However, if we compute the recursion relation, we notice that the compatibility conditions cannot be satisfied simultaneously for $c \neq 0$. For

$$c = \frac{2\sqrt{3} \pm \sqrt{48 + 192\mu}}{6}, \tag{4.110}$$

the balances $\mathcal{F}_{1,2}$ define the Puiseux expansion

$$u_{1,2} = \alpha_{1,2}(t - t_*)^{-1/2} \sum_{i=0}^{\infty} a_i (t - t_*)^i, \tag{4.111}$$

with a_3 arbitrary and the balances $\mathcal{F}_{3,4}$ define a series with logarithmic terms. However, for

$$c = \frac{-2\sqrt{3} \pm \sqrt{48 + 192\mu}}{6}, \tag{4.112}$$

the balances $\mathcal{F}_{3,4}$ are the leading terms of a Puiseux expansion of the form (4.111) and the series starting with $\mathcal{F}_{1,2}$ contain logarithmic terms. We conclude that unless $c = 0$ there is no quasimonomial transformation for which the transformed system enjoys the Painlevé property. For $c = 0$, the system can be transformed by the quasimonomial transformation T_C (with $C = \mathrm{diag}(3, 1)$) to a Painlevé integrable system. Nevertheless, for $c \neq 0$, expansions (4.111) contain valuable information concerning the existence of heteroclinic orbits in the phase space and so-called *front solutions* for the PDE. ■

4.15 The Painlevé test for the Lotka-Volterra form

In the previous section, we showed that quasimonomial transformations can be used for the Painlevé test. The Painlevé property can be tested equivalently on any member of a given equivalence class with proper modifications. Therefore, it is natural to perform the Painlevé test on the simplest representative of the class, that is, the Lotka-Volterra canonical form. Any system $S(A, B; \mathbf{x}, t)$ can be mapped by a transformation $\mathbf{x} = T_C(\mathbf{x}')$ to the canonical form $S_{LV}(M, \mathrm{I}; \mathbf{x}', t)$ where $M = BA$. Therefore, we consider the system

$$S_{LV}(M) : \dot{\mathbf{x}} = \mathbf{x}(M\mathbf{x}), \tag{4.113}$$

with $M \in \mathrm{M}_m(\mathbb{C})$.

4.15.1 Step 1: dominant balances

Following the Painlevé test, we first find all possible balances. The Lotka-Volterra system is homogeneous of degree 2. Therefore, $\mathbf{p} = -\mathbf{1}$ and the possible balances are the solutions of

$$-\boldsymbol{\alpha} = \boldsymbol{\alpha} M \boldsymbol{\alpha}. \tag{4.114}$$

As before, the different balances can be ordered by the number of vanishing components of $\boldsymbol{\alpha}$. Without loss of generality, we assume that $\alpha_i \neq 0$ for $i = 1, \dots, l$ and $\alpha_i = 0$ for all $i > l$ and we introduce the decomposition

$$M = \begin{bmatrix} M^{(1)} & M^{(2)} \\ M^{(3)} & M^{(4)} \end{bmatrix}, \tag{4.115}$$

where $M^{(1)} \in \mathrm{M}_{l,l}$. The nonvanishing components $\boldsymbol{\beta}$ of $\boldsymbol{\alpha} = (\boldsymbol{\beta}, \mathbf{0})$ are the solutions of

$$-\mathbf{1} = M^{(1)} \boldsymbol{\beta}. \tag{4.116}$$

If $M^{(1)}$ is non-singular, there is a unique solution for $\boldsymbol{\beta} = (M^{(1)})^{-1}(-\mathbf{1})$. If $M^{(1)}$ is singular, then for (4.116) to have a solution, the Fredholm condition must be satisfied. That is,

$$-\mathbf{1}.\boldsymbol{\gamma} = 0 \quad \forall\, \boldsymbol{\gamma} \in \ker(M^{(1)}), \tag{4.117}$$

According to Proposition 4.10, this last relation implies, that the system $\dot{\mathbf{y}} = \mathbf{y} M^{(1)} \mathbf{y}$ has the first integrals

$$\mathbf{y}^{\boldsymbol{\gamma}} = K \quad \forall\, \boldsymbol{\gamma} \in \ker(M^{(1)}). \tag{4.118}$$

The arbitrary constants are evaluated on the solution $\mathbf{y} = \boldsymbol{\beta} t^{-\mathbf{1}}$ to be $K = 1$. Therefore, we have

$$\boldsymbol{\beta}^{\boldsymbol{\gamma}} = 1 \quad \forall\, \boldsymbol{\gamma} \in \ker(M^{(1)}). \tag{4.119}$$

The solutions of (4.119) together with (4.116) provide the values of the leading coefficients of the balance.

4.15.2 Step 2: Kovalevskaya exponents

For a given balance $\mathcal{F}$ of order l, the Kovalevskaya exponents can be easily computed from the canonical form $S(M, \mathrm{I})$. Expression (4.97) reads now

$$K = \begin{bmatrix} \mathrm{diag}(\boldsymbol{\beta}) M^{(1)} & \mathrm{diag}(\boldsymbol{\beta}) M^{(2)} \\ 0 & \mathrm{diag}(M^{(3)}\boldsymbol{\beta} + \mathbf{1}) \end{bmatrix}. \tag{4.120}$$

Since the lower right block of K is diagonal, $(n-l)$ Kovalevskaya exponents are given by $\mathrm{diag}(M^{(3)}\boldsymbol{\beta} + \mathbf{1})$ and we have

$$\mathcal{R} = \{\mathrm{Spec}(\mathrm{diag}(\boldsymbol{\beta}) M^{(1)}), M^{(3)}\boldsymbol{\beta} + \mathbf{1}\}. \tag{4.121}$$

Moreover if $M^{(1)}$ is of rank k, then $(l-k)$ Kovalevskaya exponents identically vanish.

4.15.3 Step 3: compatibility conditions

We consider the full system and for a given dominant balance $\mathcal{F}$, we look for a Laurent series solution of the form

$$\mathbf{x} = (t - t_*)^{\mathbf{p}} \sum_{j=0}^{\infty} \mathbf{a}_j (t - t_*)^j, \tag{4.122}$$

with $\mathbf{a}_0 = \boldsymbol{\alpha}$. A lengthy computation gives a recursion relation for the coefficients $\mathbf{a}_i$ which reads

$$K\mathbf{a}_j = \mathbf{P}_j - \mathcal{E}_k(\mathbf{a}_1, \dots, \mathbf{a}_{j-1}), \tag{4.123.a}$$

$$\mathbf{P}_j = j\mathbf{a}_j - \sum_{i=1}^{j-1} \mathbf{a}_i \mathbf{P}_{j-i}, \tag{4.123.b}$$

with $\mathbf{P}_0 = \mathbf{0}$. The term $\mathcal{E}_k$ is polynomial in its arguments and can be given explicitly (Goriely, 1992). The compatibility conditions for the Kovalevskaya exponent $\rho_j \in \mathcal{R}^{(+)}$ can be easily obtained by applying the eigenvectors of K^T to (4.123.a). That is,

$$C_j = \bar{\boldsymbol{\beta}}_j.(\mathbf{P}_j - j\mathbf{a}_j - \mathcal{E}_k(\mathbf{a}_1, \dots, \mathbf{a}_{j-1})). \tag{4.124}$$

The recursion relation for $\mathbf{P}_k$ can be inverted to yield an explicit form of $\mathbf{P}_i$ as a polynomial in the terms $\mathbf{a}_i$:

$$\mathbf{P}_k = \sum_{\{i_j\}} c(\{i_j\}) \prod_{j=1}^{k} \mathbf{a}_j^{i_j}, \tag{4.125.a}$$

$$c(\{i_j\}) = \frac{k\beta!(-)^{\beta}}{\prod_{j=1}^{k} i_j!}, \tag{4.125.b}$$

where $\beta = -1 + \sum_{j=1}^{k} i_j$ and the sum over $\{i_j\}$ is taken with the condition $\sum_{j=1}^{k} i_j j = k$. The important feature of this relation is that $\mathbf{P}_k$ is weight-homogeneous of degree k with respect to the weight $g_i = i$. That is, the scaling $\mathbf{a}_i \to \epsilon^i \mathbf{a}_i$ is such that

$$\mathbf{P}_k(\epsilon\mathbf{a}_1, \epsilon^2\mathbf{a}_2 \dots, \epsilon^k\mathbf{a}_k) = \epsilon^k \mathbf{P}_k(\mathbf{a}_1, \dots, \mathbf{a}_k). \tag{4.126}$$

Let $\mathbb{K}$ be a field of constants. A finite subset $\{e_1 \dots, e_n\}$ of a complex vector space E is $\mathbb{K}-$*independent* if the equality $\sum_{i=1}^{n} k_i e_i = 0$, where $k_i \in \mathbb{K}$, implies $k_1 = \dots = k_n = 0$ (see Definition 5.1).

Lemma 4.4 *If the set, $\mathcal{R}^{(+)}$, of all positive Kovalevskaya exponents for a balance $\mathcal{F}$ is $\mathbb{N}$-independent, then the compatibility conditions are satisfied.*

This lemma is very useful in practice since it ensures the existence of a formal Laurent solution from the Kovalevskaya exponents.

Proof. For all $k \le \max(\mathcal{R}^{(+)})$, the term $\mathcal{E}_k$ vanishes identically and the recursion relation reduces to

$$K\mathbf{a}_k = \mathbf{P}_k. \tag{4.127}$$

Owing to the weight-homogeneity of P_j and the $\mathbb{N}$-independence of $\mathcal{R}^{(+)}$, for all $k \le \max(\mathcal{R}^{(+)})$, we have,

$$\mathbf{P}_k = k\mathbf{a}_k. \tag{4.128}$$

Therefore, we conclude that

$$\mathbf{a}_{\rho_i} = \boldsymbol{\beta}^{(i)} \qquad \forall\ \rho_i \in \mathcal{R}^{(+)}, \tag{4.129.a}$$

$$\mathbf{a}_k = 0 \quad \forall\ k \notin \mathcal{R}^{(+)}, \qquad \forall\ k < \max(\mathcal{R}^{(+)}), \tag{4.129.b}$$

where, as usual, $\boldsymbol{\beta}^{(i)}$ is the eigenvector of K of eigenvalue ρ_i. The compatibility conditions are satisfied identically for all positive Kovalevskaya exponents. □

This lemma allows us to verify the compatibility conditions without actually explicitly computing the coefficients of the Laurent series. The conditions on the Kovalevskaya exponents are usually not necessary since the compatibility conditions are satisfied due to the fact that $\mathbf{P} - r\mathbf{a}_r$ vanishes identically.

4.16 Transformation of singularities

Despite the efficiency of the Painlevé test to detect new integrable systems, an important question remained unanswered. The discovery of two-degree-of-freedom integrable Hamiltonian systems with rational Kovalevskaya exponents was at the origin of the weak-Painlevé conjecture (Grammaticos *et al.*, 1984). The main idea behind the weak-Painlevé conjecture is to find sufficient conditions on the singularities in order for the system to be Liouville integrable (see Section 3.10). The question of whether these weak-Painlevé systems can be mapped onto a full Painlevé system is discussed in detail in Hietarinta *et al*. (1984) where it is found that for certain systems, such a transformation exists. However, in general, the weak-Painlevé conjecture cannot be reduced to the Painlevé test. Therefore, considered as a criterion of integrability, it remained of heuristic value.

Next we show here that the new-time transformations N_β, introduced in Section 4.3, can be used to transform the Kovalevskaya exponents and the dominant behaviors in a simple way. It is therefore possible to modify the type of singularities that the solutions exhibit.

4.16.1 New-time transformation of local series

Let $S(A, B; \mathbf{x}, t) \in \mathcal{Q}_n$ be a quasimonomial system and consider the transformation $N_{\boldsymbol{\beta}} : dt = \mathbf{x}^{\boldsymbol{\beta}} d\widetilde{t}$. This transformation maps S to $N_{\boldsymbol{\beta}}(S) = \widetilde{S}(A, B \oplus \boldsymbol{\beta}; \mathbf{x}, \widetilde{t})$. For a given balance $\mathcal{F}$ of system S, we can compute the corresponding balance $\widetilde{\mathcal{F}}$ for system $\widetilde{S}$. Since the leading exponents and the Kovalevskaya exponents are expressed in terms of $\{A, B\}$ through relations (4.77.a)-(4.77.b), (4.78) and (4.97), and the corresponding exponents for the transformed system $N_{\boldsymbol{\beta}}(S) = \widetilde{S}$ are

$$\widetilde{\mathbf{p}} = \frac{\mathbf{p}}{c}, \quad \widetilde{q}^{(i)} = \frac{q^{(i)}}{c}. \tag{4.130}$$

The transformation of Kovalevskaya exponents has the same form. That is,

$$\widetilde{\rho} = \begin{cases} -1 \text{ for one Kovalevskaya exponent } \rho = -1, \\ \dfrac{\rho}{c} \quad \text{otherwise,} \end{cases} \tag{4.131}$$

with $c = \mathbf{p}.\boldsymbol{\beta} + 1$ and $c \neq 0$. A balance $\mathcal{F}$ is dominant if it represents the behavior of the solution in the limit $t \to t_*$. In terms of the non-dominant exponents, the dominance condition is given by $q^{(i)} > 0 \ \forall\ i > 0$. After a new-time transformation, we must verify that the balance is still dominant, that is, $\widetilde{q}^{(i)} > 0 \ \forall\ i$. Let $\mathcal{F}^{(+)}$ be the set of dominant balances and $\mathcal{F}^{(-)}$ the set of non dominant balances, that is, $\mathcal{F}^{(-)}$ is the set of solutions $\{\boldsymbol{\alpha}, \mathbf{p}\}$ with non-dominant exponents $q^{(i)} < 0 \ \forall\ i$. Two different situations can be distinguished.

- **Case 1:** $c > 0$**.** Then, $q^{(i)} > 0$, implies $\widetilde{q}^{(i)} > 0$ and the dominant balance $\mathcal{F}$ remains dominant after a new-time transformation. That is,

$$N_{\boldsymbol{\beta}}(\mathcal{F}^{(+)}) = \widetilde{\mathcal{F}}^{(+)}. \tag{4.132}$$

- **Case 2:** $c < 0$**.** Then, $q^{(i)} > 0$ implies $\widetilde{q}^{(i)} < 0$ and the dominant balance $\mathcal{F}$ does not remain dominant after a new-time transformation, that is, dominant and non-dominant behaviors are exchanged in the transformation. That is,

$$N_{\boldsymbol{\beta}}(\mathcal{F}^{(+)}) = \widetilde{\mathcal{F}}^{(-)}. \tag{4.133}$$

The new-time transformation acts on the Newton polyhedron as a translation parallel to the vector $\boldsymbol{\beta}$ and preserves the polygon faces. Some faces of the polygon correspond to dominant balance $\mathcal{F}^{(+)}$ while others correspond to non-dominant balance $\mathcal{F}^{(-)}$. Lemma 4.8 guarantees that a face of the polygon remains a face after a new-time transformation. Moreover, all inequalities, like the dominance condition, are conserved if $c > 0$ and are reversed if $c < 0$.

In terms of singularity analysis, there is a subtle point related to the new-time transformation with $c < 0$. There is a possibility that the transformation introduces spurious logarithms. The time t is related to the new-time $\widetilde{t}$ by

$$t = \int \mathbf{x}^{\boldsymbol{\beta}}(t - t_*)d\widetilde{t}. \tag{4.134}$$

Therefore, if $\text{res}_{t_*}(\mathbf{x}^{\boldsymbol{\beta}}(t - t_*)) \neq 0$, the logarithmic contributions enter in time t. If $c > 0$, we have $\text{res}_{t_*}(\mathbf{x}^{\boldsymbol{\beta}}(t - t_*)) = 0$. However, for $c < 0$, we have to check, case by case, that the residue vanishes identically.

Example 4.15 Continuation of Example 4.11. As an example of this phenomenon, we consider again system (4.11) to which we apply two new-time transformations, one with $\boldsymbol{\beta} = (1, 1)$ and the other with $\boldsymbol{\beta} = (-1, -1)$. In the first case, all dominant faces (Γ_2, Γ_3) remain dominant after the transformation. In the second case, every face of the Newton polygon becomes dominant after the transformation (see Figure 4.6)). ■

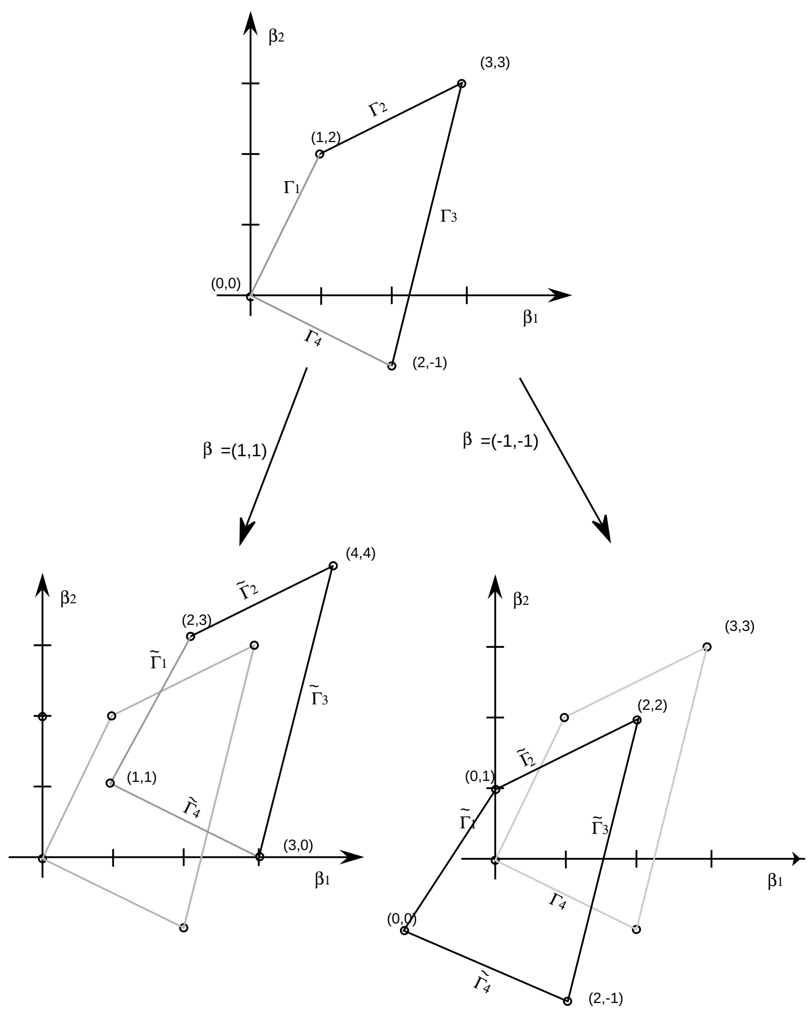

Figure 4.6: Two different transformations N_β acting on the Newton polygon. The dominant faces are in black and the non-dominant faces are dashed. Note that after the transformation with $\boldsymbol{\beta} = (-1, -1)$, all faces are dominant. That is, new dominant balances are created in the transformation.

We can now summarize the previous results.

Proposition 4.17 *Consider a system* $S: \ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \in \mathcal{Q}_n$ *and assume it has a Puiseux solution*

$$\mathbf{x} = (t - t_*)^{\mathbf{p}} \sum_{i=0}^{\infty} \mathbf{a}_i (t - t_*)^{i/s}. \tag{4.135}$$

Let $c = \mathbf{p}.\boldsymbol{\beta} + 1 > 0$*, then the solution of system* $\widetilde{S} = N_\beta(S)$ *has the series expansion*

$$\mathbf{x} = (\widetilde{t} - \widetilde{t}_*)^{\widetilde{\mathbf{p}}} \sum_{i=0}^{\infty} \widetilde{\mathbf{a}}_i (\widetilde{t} - \widetilde{t}_*)^{i/\widetilde{s}}, \tag{4.136}$$

where $\widetilde{\mathbf{p}} = \mathbf{p}/c$.

4.16.2 The weak-Painlevé conjecture

Consider an Hamiltonian system with kinetic diagonal parts (see Section 3.10):

$$H = \frac{1}{2}(p_1^2 + \ldots + p_n^2) + V(x_1, \ldots, x_n). \tag{4.137}$$

Definition 4.11 (Grammaticos *et al.*, 1983; Ramani *et al.*, 1989) The Hamiltonian system $H = \frac{1}{2}(p_1^2 + \ldots + p_n^2) + V(x_1, \ldots, x_n)$ has the *weak-Painlevé property* if all its solutions can be expanded in Puiseux series whose Kovalevskaya exponents are such that their denominator divides the smallest common multiple of the leading exponents denominators.

Let g be the smallest common multiple of the denominators of the leading exponents. That is, let $p_i = p_i'/r$ where r is the smallest number such that $p_i' \in \mathbb{Z}$ for all i. Therefore, the system has the weak-Painlevé property if the solutions can be expanded in Puiseux series and if all Kovalevskaya exponents ρ_i (for all dominant balances) are such that $r\rho_i \in \mathbb{Z}$. In particular, if $r = 1$, the system satisfies the usual Painlevé test. The weak-Painlevé conjecture can be phrased as follows.

> **Conjecture:** If the Hamiltonian system (4.137) has the weak-Painlevé property, then it is Liouville integrable.

For non-Hamiltonian systems, the Weak-Painlevé property is simply the existence of Puiseux series.

Definition 4.12 The system $S: \ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has the *weak-Painlevé property* if all its solutions can be expanded in Puiseux series.

Making use of the above transformation rules, it clearly appears that for a given balance $\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$ with non integer Kovalevskaya exponents of weak-Painlevé type, a well chosen new-time transformation allows us to map this balance onto a Painlevé-type balance with integer Kovalevskaya exponents. The origin of the natural denominator restriction is directly connected to this kind of transformation. If we choose $c = 1/r$ in relations (4.130)-(4.131), then in new set of variables, both dominant exponents and Kovalevskaya exponents are integers; that is, the Puiseux series is mapped to a Laurent solution for the new system. The mapping transforming a weak-Painlevé solution to a Laurent solution will be referred to as a *regularization*. This regularization proceeds in two steps. First, we compute the new-time transformation mapping balances with rational Kovalevskaya exponents and non-dominant exponents to balances with integer valued Kovalevskaya exponents. The new vector field is generally not rational, and the Painlevé test cannot be applied directly. This is why we perform a quasimonomial transformation to the Lotka-Volterra form where the occurrence of a logarithmic branch point can be readily tested.

The main obstruction to a general regularization for all systems possessing the weak-Painlevé property is that the system must satisfy the extended Painlevé test if all balances are compatible. Therefore, all transformed balances have to be checked after a new-time transformation and it might not be possible to regularize all different balances with a unique new-time transformation.

Only a few systems are actually known to satisfy the weak-Painlevé test; among others are the Fokas-Lagerstrom Hamiltonian, the Holt Hamiltonian (Holt, 1982) and the class V_n defined in (4.138). Note that the role of algebraic singularities was already emphasized by Painlevé himself (Painlevé, 1973, p. 135)

> Enfin, en même temps que les équations à intégrale uniforme, il est naturel de considérer les équations dont l'intégrale générale est une fonction *à nombre fini de branches.* L'étude de ces équations fait intervenir, comme équations intermédiaires bien remarquables par elles-mêmes, les *equations dont l'intégrale générale n'acquiert que n valeurs autour des points critiques mobiles.* [1]

The underlying hope of the weak-Painlevé conjecture is that, at least for a given class of systems (for instance, polynomial Hamiltonians), such a connection could be established. Unfortunately, we show that this is not the case by exhibiting a system with the weak-Painlevé property which is not integrable since its dynamics is chaotic. We now show that, at least for the above systems, such a regularization can be achieved.

Example 4.16 Regularization of a polynomial potential. Dorizzi *et al.* (1983) presented a new class of integrable homogeneous Hamiltonian system,

$$H = 1/2(p_x^2 + p_y^2) + V_n, \tag{4.138.a}$$

$$V_n = \sum_{k=0}^{[n/2]} 2^{n-2k} C_{n-k}^k x^{2k} y^{n-2k}, \tag{4.138.b}$$

and computed explicitly the second first integral, quadratic in velocities. For $n > 4$ in (4.138), all potentials V_n give rise to a weak-Painlevé system with a dominant behavior $x, y \sim (t - t_*)^{-2/(n-2)}$. We present here the regularization of V_5. The other cases can be treated along the same lines as those presented here. For V_5, the Hamiltonian reads

$$H = 1/2(p_x^2 + p_y^2) + 32y^5 + 32x^2y^3 + 6x^4y. \tag{4.139}$$

This system is weight-homogeneous and the dominant exponents are $\mathbf{p} = (-2/3, -2/3, -5/3, -5/3)$ where the variables in $\mathbf{p}$ are ordered according to $\mathbf{x} = (x, y, p_x, p_y)$. There are two sets of different dominant balances of order 2 and 4. The first set is characterized by $\alpha_1 = \alpha_3 = 0$ and includes three different balances with Kovalevskaya exponents $\mathcal{R}^{(1)}$. The second set is of order 4 and includes ten different balances with Kovalevskaya exponents $\mathcal{R}^{(2)}$. The Kovalevskaya exponents are, respectively,

$$\mathcal{R}^{(1)} = \{-1, \frac{1}{3}, 2, \frac{10}{3}\}, \quad \mathcal{R}^{(2)} = \{-1, \frac{-5}{3}, \frac{10}{3}, 4\}. \tag{4.140}$$

The natural denominator for both balances in (4.140) is $r = 3$. Therefore, the obvious choice for a regularization of (4.138) is $c = 1/3$. The new-time transformation $N_{\boldsymbol{\beta}}$ with $\boldsymbol{\beta} = (0, -1, 0, 0)$ leads to a new set of equations which satisfy the Painlevé test with $\mathbf{p} = (-2, -2, -5, -5)$,

$$\widetilde{\mathcal{R}} = \{-1, 1, 6, 10\}, \text{ and } \widetilde{\mathcal{R}} = \{-1, -5, 10, 12\}. \tag{4.141}$$

The system $S(A, B \oplus \boldsymbol{\beta}; \mathbf{x}, \widetilde{t})$ reads

$$\frac{dx_1}{d\widetilde{t}} = x_2^{-1} x_3, \tag{4.142.a}$$

$$\frac{dx_2}{d\widetilde{t}} = x_2^{-1} x_4, \tag{4.142.b}$$

$$\frac{dx_3}{d\widetilde{t}} = x_2^{-1} \left(-160x_1^4 - 24x_1^3 x_2 - 64x_1 x_2^2\right), \tag{4.142.c}$$

$$\frac{dx_4}{d\widetilde{t}} = x_2^{-1} \left(-6x_1^4 - 96x_1^2 x_2^2\right), \tag{4.142.d}$$

and passes the Painlevé test in the variables $\{\mathbf{x}, \widetilde{t}\}$. ■

[1] Finally, together with equations with single-valued solutions, it is natural to consider equations for which the general solution exhibits *finite branching.* The analysis of these equations introduce new remarkable equations for which the general solution only exhibits movable algebraic branch points.

For a given Hamiltonian system, there is a close relationship between the new-time transformations and canonical transformation on the Hamiltonian itself. The re-parametrization, $d\widetilde{t} = Fdt$ for an Hamiltonian H_0, is equivalent to a duality transformation between the Hamiltonians $H = H_0 - gF$ (where g is a free coupling-constant) and $G = H_0/F - h/F$ (Hietarinta *et al.*, 1984). The set of first integrals of H and G is identical as a result of the invariance of the integrals with respect to the time parametrization. Here, we use the new-time transformation in a more general setting. Hence, we apply the transformation at the level of the equations and do not compute the dual Hamiltonian. This procedure can therefore be applied to non-Hamiltonian systems and provides a systematic way for determining the new-time transformation required for the regularization.

Example 4.17 The soft hyperbolic billiard. Consider the Hamiltonian

$$H = 1/2(p_x^2 + p_y^2) - \frac{1}{\gamma}(x^2y^2)^\gamma, \tag{4.143}$$

with $\gamma > 1/2$. This system has been extensively studied from the point of view of dynamical systems (Savvidy, 1983; Mikolaevskii & Schur, 1983; Chang, 1984; Carnegie & Percival, 1984; Hansen, 1992; Jaroensutasinee & Rowlands, 1994) and was shown to reduce to a billiard with hyperbolic walls for $\gamma \to \infty$. The complete stochasticity of this asymptotic system can then be demonstrated. For finite values of γ, the system has soft walls and also exhibits chaotic dynamics. This system has many interesting features. First, notice that for $\gamma = -1/3$, it reduces to the Fokas-Lagerstrom potential which can be regularized (see Exercise 4.23). Second, in Chapter 5, we prove that for $2\gamma \in \mathbb{Z} \setminus \{-2, 0, +2\}$ there is no other analytical first integral than H itself. To do so, we use the Ziglin theory of nonintegrability for Hamiltonian system (Section 5.3.4).

However, despite these observations that suggest nonintegrability, the system has the weak-Painlevé property for a countable infinity of values of γ. To see this explicitly, let $\mathbf{x} = (x, y, p_x, p_y)$. There is a unique dominant balance with exponent $\mathbf{p} = (-2/d, -2/d, -(2+d)/d), -(2+d)/d)$ where $d = 4\gamma - 2$. The corresponding Kovalevskaya exponents are

$$\mathcal{R} = \{-1, \frac{2(d+2)}{d}, \frac{(d+4) \pm \sqrt{d^2 - 8d - 16}}{2d}\}. \tag{4.144}$$

There are infinitely many values of γ leading to rational Kovalevskaya exponents. However, not all these values lead to systems with the weak-Painlevé property since the compatibility conditions might not be satisfied. Nevertheless, there exists a set of values of γ for which the three positive Kovalevskaya exponents are rational and $\mathbb{N}$-independent. The first such values are

$$\gamma \ \in \ \{35/12, 117/40, 165/56, 425/144, \dots\}. \tag{4.145}$$

We can now use Lemma 4.4 to conclude that the system has the weak-Painlevé property for these values of γ. None of these systems can be regularized by a new-time transformation. A numerical investigation of system (4.143) shows that the Poincaré section exhibits a large chaotic region incompatible with integrability. This analysis is confirmed by studies on the potential (4.143) showing the existence of stable periodic orbits surrounded by huge chaotic regions (Dalhqvist & Russberg, 1990).

We conclude that system (4.143) with values (4.145) is a counterexample of the weak-Painlevé conjecture. Nevertheless, it is obvious that, at least for a given class of systems, integrability is linked to some algebraic singularity structures. ■

4.16.3 New integrable systems

The new-time transformation also can be used to find new integrable systems.

Example 4.18 The two-dimensional Lotka-Volterra system. Consider the system

$$\dot{x}_i = \lambda_i x_i + x_i M_{ij} x_j, \qquad i = 1, 2, \tag{4.146}$$

with

$$M = \begin{bmatrix} -1 & a \\ b & -1 \end{bmatrix}$$

We study the case $\lambda_1 = \lambda_2 = 0$ or equivalently $\lambda_1 = \lambda_2$ (the linear terms can be removed by the transformations $X_i = e^{\lambda t} x_i$, $dT = e^{-\lambda t} dt$). The case $\lambda_1 = \lambda_2 = 0$ was already studied in Section 3.8 (Equation (3.83)) where is was shown that the Kovalevskaya exponents are irrational for most values of the parameters. The three balances are

$$\mathcal{F}_1 = \{\boldsymbol{\alpha} = (1, 0),\ \mathbf{p} = (-1, -1)\}, \tag{4.147.a}$$

$$\mathcal{F}_2 = \{\boldsymbol{\alpha} = (0, 1),\ \mathbf{p} = (-1, -1)\}, \tag{4.147.b}$$

$$\mathcal{F}_3 = \{\boldsymbol{\alpha} = (\frac{1+b}{1-ab}, \frac{1+a}{1-ab}),\ \mathbf{p} = (-1, -1)\}. \tag{4.147.c}$$

with Kovalevskaya exponents $\mathcal{R}_1 = \{-1, 1+b\}$, $\mathcal{R}_2 = \{-1, 1+a\}$ and $\mathcal{R}_3 = \{-1, \frac{(b+1)(a+1)}{(ab-1)}\}$.

For arbitrary values of a and b such that $ab \neq -1$, the system can be regularized by a NTT with $\boldsymbol{\beta} = (\frac{b(1+a)}{1-ab}, \frac{a(1+b)}{1-ab})$, followed by a QMT with matrix $C = \begin{bmatrix} 1/d & a/d \\ b/d & 1/d \end{bmatrix}$ and $d = ab - 1$. After these two transformations, we obtain the system

$$\frac{dx'_i}{d\widetilde{t}} = -{x'}_i^2, \qquad i = 1, 2. \tag{4.148}$$

This system is trivially integrable with Kovalevskaya exponents $\mathcal{R} = \{-1, -1\}$ and furthermore it has the Painlevé property. Therefore, we conclude that system (4.146) satisfies the Painlevé test in the variables $\{\mathbf{x}', \widetilde{t}\}$ if $\lambda_1 = \lambda_2$. The corresponding first integral in the original variables is

$$I = x_1^{\frac{b+1}{d}} x_2^{-\frac{1+a}{d}} \left[(a+1)x_2 - (b+1)x_1\right], \tag{4.149}$$

and the explicit form of the NTT is

$$\widetilde{t} - \widetilde{t}_0 = \int_{(x_1(0))^{-b/d}(x_2(0))^{-1/d}}^{x_1^{-b/d} x_2^{-1/d}} d\mu \left[\left(I + \frac{\mu^{a+1}}{(a+1)} \right)^{\frac{b(a-2)}{(b+1)}} \mu^{a(b-1)} \right], \tag{4.150}$$

where $d = 1 - ab$ and $x_i(0)$ corresponds to the initial conditions. We conclude that even in the case of non-rational Kovalevskaya exponents, the NTT can be used not only to detect integrable systems but also to integrate them. ■

Example 4.19 A three-dimensional Lotka-Volterra system. Consider the three-dimensional Lotka-Volterra system (2.23)

$$\dot{x}_1 = x_1(\lambda_1 + cx_2 + x_3), \tag{4.151.a}$$

$$\dot{x}_2 = x_2(\lambda_2 + x_1 + ax_3), \tag{4.151.b}$$

$$\dot{x}_3 = x_3(\lambda_3 + bx_1 + x_2). \tag{4.151.c}$$

Our aim here is not to achieve a complete study of system (4.151) but rather to show how new solutions can be found using the Painlevé test in conjunction with the new-time transformations. If $abc + 1 = 0$, system (4.151) is algebraically degenerate as defined in Section 4.8 and can be mapped onto a two-dimensional Lotka-Volterra system equivalent to (4.146). Consider the generic case, $abc + 1 \neq 0$. Once again, to find new solutions, we take $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and remove the linear diagonal terms by the transformations ($X_i = e^{\lambda t} x_i$, $dT = e^{-\lambda t} dt$). If $a = 1,\ b > 0,\ c > 1$, we obtain a single balance of order 3 defined by

$$\mathbf{p} = (-1, -1, -1), \qquad \mathcal{R} = \{-1, \frac{c}{\delta}, \frac{1 - c + bc}{\delta}\}, \tag{4.152}$$

where $\delta = (1 + bc)$. If $b, c \in \mathbb{N}$, then, we can regularize this system by a NTT with $\beta = (-bc/\delta, 0, 0)$. The QMT with

$$C = \begin{bmatrix} \delta & 0 & 0 \\ \delta - 1 & 1 & 0 \\ \delta - 1 & 0 & 1 \end{bmatrix}, \tag{4.153}$$

gives a new three-dimensional system which satisfies the Painlevé test, the first integrals of which are

$$I_1 = \left(c x_1^{-1/\delta} x_2^{1/\delta} x_3^{1/\delta} - x_1^{bc/\delta} x_2^{-bc/\delta} x_3^{c/\delta} \right) e^{-\frac{c\lambda}{\delta} t} \tag{4.154.a}$$

$$I_2 = e^{\lambda t} x_1^{1/\delta} x_2^{-bc/\delta} x_3^{-c\delta} - \int^{x_1^{b/\delta} x_2^{-b/\delta} x_3^{1/\delta} e^{\lambda t}} dv \left(\frac{v^c}{c} + I_1 \right)^{b-1}, \tag{4.154.b}$$

Following the same procedure, other integrable cases can be detected (Goriely, 1992). ■

4.17 Exercises

4.1 Find matrices A and B for the system

$$\dot{x}_1 = x_1^2 + x_1 x_2, \tag{4.155.a}$$
$$\dot{x}_2 = x_1^2 + x_1 - x_1 x_2. \tag{4.155.b}$$

What is the canonical Lotka-Volterra form of this system? Transform and then solve this system by a new time transform, $d\tilde{t} = x_1 dt$.

4.2 Find the matrices A and B for the system

$$\dot{x}_1 = -3x_1 + \frac{1}{2}x_1 x_2 + 2x_1 x_3 - 5x_2 x_3, \tag{4.156.a}$$
$$\dot{x}_2 = x_1 x_2 - x_2 x_3 - x_2, \tag{4.156.b}$$
$$\dot{x}_3 = x_3(2x_1 + 7x_2) - x_1. \tag{4.156.c}$$

What is the quasimonomial transformation that maps this system to a Lotka-Volterra system?

4.3 Consider an autonomous linear system in n dimensions:

$$\dot{\mathbf{x}} = L\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n. \tag{4.157}$$

What are the dimensions n, m of the corresponding matrices A and B?

4.4 Consider a quasimonomial system $S(A, B; \mathbf{x}, t)$ and assume that the matrix $M = BA$ is triangular. Show that the original system can be solved in closed form. What is the solution?

4.5 In biology and chemistry, models often involve rational vector fields (typically, to express the saturation of a compound or a population when its concentration or size increases). These systems can also be cast in the quasimonomial formalism. To do so, we introduce the denominators of these functions as new auxiliary variables. These auxiliary variables obey polynomial differential equations and the new system (the old and auxiliary variables) is a quasimonomial system. For instance, consider the equation modeling the concentration of an allosteric enzyme (Hernández-Bermejo & Fairén, 1995)

$$\dot{x} = -x\frac{a + bx}{c + x + dx^2}. \tag{4.158}$$

Introduce the new variable $y = (c + x + dx^2)^{-1}$ and show that it obeys a polynomial differential equation. Find the matrices A and B that characterize this system.

4.6 Transform the Morse oscillator,

$$\ddot{x} = -2d\alpha e^{\alpha x}(1 - e^{-\alpha x}), \tag{4.159}$$

into a quasimonomial system by first writing the equation as a system of two first order ODEs, and second by introducing $y = e^{\alpha x}$ as an extra variable. Find the corresponding canonical Lotka-Volterra form (Hernández-Bermejo & Fairén, 1995).

4.7 Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Find all admissible functions $\mathbf{f}$ such that, after a suitable embedding, the system can be written as a quasimonomial system (see Kerner (1981) for similar considerations).

4.8 Find the Newton polygon of the system given in Problem 1. Compute the transformed Newton polygon after the new-time transformation.

4.9 Build the matrices A and B and Newton's polyhedron associated with the vector field

$$\dot{x}_i = \sum_{j=1,2,3} \alpha_{ij} x_j + x_i \sum_{j=1,2,3} M_{ij} x_j, \qquad i = 1, 2, 3. \tag{4.160}$$

4.10 Find the vector field in $\mathbb{R}^3$ whose Newton's polyhedron is a cube with a given integer-valued side length.

4.11 Show that the Newton polyhedron of a weight-homogeneous system has only one face. Find the relationship between the intersection of the support hyperplane of the polyhedron with the axes and the weights.

4.12 Consider a quasimonomial system $S(A, B, \mathbf{x}, t)$. Generalize Proposition 4.9 to test for the existence of first integrals of the form $I = \mathbf{x}^{\mathbf{m}} \exp(\mathbf{n}.\mathbf{x})$ and $I = \mathbf{x}^{\mathbf{m}}(\mathbf{n}.\mathbf{x})$. To do so, introduce new variables $y_i = e^{x_i}$ in the first case and $y = \mathbf{n}.\mathbf{x}$ in the second case.

4.13 A three-dimensional vector field is *bi-Hamiltonian* if it admits two first integrals and the if Jacobian J at every fixed point is a *Poisson matrix*Poisson matrix—textit, that is, there exists $T \in GL(3, \mathbb{C})$ such that $J = -T^{-1}JT$ (Planck, 1996b; Planck, 1996a). Show that all bi-Hamiltonian Lotka-Volterra systems have a quasimonomial first integral.

4.14 The two-dimensional system

$$\dot{x}_1 = x_1(\lambda - ax_1^2 - b(x_2^2 + 2x_1x_2) + 2x_2^3), \tag{4.161.a}$$
$$\dot{x}_2 = x_2(\lambda - bx_2^2 - a(x_1^2 - 2x_1x_2) - 2x_1^3), \tag{4.161.b}$$

admits an algebraic limit cycle. *(i)* Find the matrices A, B. *(ii)* Show that the system admits a quadratic second integral J. *(iii)* Apply the methods of Section 2.5.1 to find a first integral. *(iv)* Describe the dynamics and show that $J = 0$ is a limit cycle.

4.15 Consider the system (Rosen, 1976)

$$\dot{x}_1 = x_1(a_2b_3x_3(x_2 - \bar{x}_2) - a_3b_2x_2(x_2 - \bar{x}_3)), \tag{4.162.a}$$
$$\dot{x}_2 = x_2(a_3b_1x_1(x_3 - \bar{x}_3) - a_1b_3x_3(x_1 - \bar{x}_1)), \tag{4.162.b}$$
$$\dot{x}_3 = x_3(a_1b_2x_2(x_1 - \bar{x}_1) - a_2b_1x_1(x_2 - \bar{x}_2)), \tag{4.162.c}$$

where $\bar{\mathbf{x}}$ is a fixed point. *(i)* Compute the matrices A, B for this system. *(ii)* Show the existence of a linear first integral $I_1 = \mathbf{n}.\mathbf{x}$. *(iii)* Use the results of Exercise 4.12 to show the existence of a first integral of the form $I_2 = \mathbf{x}^{\mathbf{m}} \exp(\mathbf{n}.\mathbf{x})$. *(iv)* Find the conditions on the parameters so that the system admits positive periodic solutions. *(v)* Use these two first integrals to analyze the dynamics of the system.

4.16 For Example 4.9, use the Jacobi last multiplier theorem to compute the second first integral in the case where there exists a first integral and a multiplier; that is, when either $a+b+2 = 0$, or $a^2 - a - ab - b + 1 + b^2 = 0$.

4.17 Consider the May-Leonard system with diagonal linear terms

$$\dot{x}_i = x_i(\lambda_i + \sum_{j=1}^{3} M_{ij}x_j), \qquad i = 1, 2, 3, \tag{4.163}$$

where M is given by (4.66). Compute the conditions on the parameters $\boldsymbol{\lambda}, a, b$ for the existence of a Jacobi multiplier.

4.18 Consider a three-dimensional system and assume it admits both a quasimonomial first integral, $I = \mathbf{x}^{\mathbf{m}}$, and a quasimonomial Jacobi multiplier, $G = \mathbf{x}^{\boldsymbol{\alpha}}$. Compute the general form of the second first integral.

4.19 Use the previous results to find an example of a bi-Hamiltonian Lotka-Volterra system.

4.20 Some authors have relaxed the Painlevé test by allowing rational dominant exponents $\mathbf{p}$ and integer Kovalevskaya exponents. This test is known as the *quasi-Painlevé test* (Steeb *et al.*, 1985; Steeb & Euler, 1988). Show that if a system S passes the quasi-Painlevé test then there exists a quasimonomial transformation T_C such that $T_C(S)$ passes the Painlevé test. Use the explicit form of the transformation of the leading exponents in a quasimonomial transformation to compute the transformation.

4.21 Consider a weight-homogenous system $S(A, B; \mathbf{x}, t)$ and assume it has a dominant balance such that all Kovalevskaya exponents are strictly negative. Show that there exists a new-time transformation $N_{\boldsymbol{\beta}}$, mapping the system to a new system, $N_{\boldsymbol{\beta}}(S)$, with a dominant balance with strictly positive Kovalevskaya exponents (except for $\rho_1 = -1$).

4.22 Consider the Chazy equation

$$\frac{d^3x}{dt^3} = -\Big(\frac{16+4a}{7+a}\Big)\Big(\frac{dx}{dt}\Big)^2 + 2x\frac{d^2x}{dt^2}. \tag{4.164}$$

(i) Find the values of a such that the Kovalevskaya exponents are rational. *(ii)* For each of these values compute the Kovalevskaya exponents. *(iii)* For the values of a such that all the Kovalevskaya exponents are negative, find a new-time transformation with $c = -1$ mapping the equation to an equation with 2 positive Kovalevskaya exponents. Test the Painlevé property on these new systems.

4.23 Regularization. Consider the Fokas-Lagerstom Hamiltonian (Hietarinta *et al.*, 1984)

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{3}{2}(xy)^{-2/3}. \tag{4.165}$$

(i) Show that it has a second first integral given by $I = (x_3x_4)^2(x_3x_2 - x_4x_1) + (x_1x_2)^{-2/3}(x_3x_2 + x_4x_1)$. *(ii)* Write the corresponding matrix A and B for $S(A, B, \mathbf{x}, t)$ and use them to compute the dominant balances and the Kovalevskaya exponent. *(iii)* Show that the natural denominator is $r = 5$ and apply a new-time transformation $N_{\boldsymbol{\beta}}$ with $\beta = (2/3, 2/3, 0, 0)$ to obtain a new system $S(A, B \oplus \boldsymbol{\beta}, \mathbf{x}, \tilde{t})$. *(iv)* Show that a after a quasimonomial transformation with $C = \mathrm{diag}(1/2, 1/2, 1/2, 1/2)$, the system $T_C(S(A, B \oplus \beta; \mathbf{x}, \widetilde{t}))$ passes the Painlevé test.

Chapter 5:

Nonintegrability

"When, however, one attempts to formulate a precise definition of integrability, many possibilities appear, each with a certain intrinsic theoretic interest"
Birkhoff

Form the previous chapters, it clearly appears that a universal notion of integrability cannot be obtained. This is mainly due to a confusion between Hamiltonian theory, dynamical systems approach and singularity analysis. Each field has a different definition of integrability relevant within the theory. The difficulty arises when we try to establish possible relationships between different fields. The main problem discussed in this chapter is that of finding a connection, if any, between the Painlevé property and other notions of integrability which could be used to effectively build the solutions or gain some global knowledge on the dynamics in phase space. For instance, while it is widely believed that the Painlevé property is incompatible with chaotic motions, there is to date no rigorous proof of this simple statement. It is also known that Liouville integrability is not directly related to the Painlevé property despite some intriguing connections. More generally, we need a simple test for the existence or non-existence of first integrals in a given function space (polynomial, rational, algebraic,...) and it is the purpose of this chapter to show that singularity analysis provides it.

In Chapter 2, algebraic integrability for an n-dimensional systems of ODEs with rational vector fields was defined as the existence of $(n-1)$ algebraic first integrals. This notion of integrability is very strong. For instance, Liouville integrability for n-degree-of-freedom Hamiltonians only requires the existence of n first integrals; the remaining $(n-1)$ angle variables, expressed as closed 1-forms, are not, in general, algebraic or even single-valued first integrals. Liouville integrability is therefore a much weaker statement on singularities than algebraic integrability. This explains why singularity analysis is not used to prove Liouville integrability for general Hamiltonians. However, the notion of algebraic integrability constrains solutions of the systems in such a way that global statements on the meromorphicity of the solutions are possible. In turn, the lack of local single-valuedness can be used to prove global nonintegrability of a system.

Another fundamental problem of integrability is the notion of partial integrability, that is, when the number of first integrals is less than the number required for complete integration. Singularity analysis has been a successful tool for finding integrable systems. Many new Hamiltonian and non-Hamiltonian systems have been built this way. However, most Hamiltonian systems, although Liouville integrable, cannot be detected by singularity analysis. In the same way, most dynamical systems admit a few invariants related to physical conservation laws, but only a handful of them will admit enough first integrals to build effectively a complete analytical global solution. More generally, there is no decision procedure to test the simple question: is there a polynomial, rational or logarithmic first integral for a given systems of ODEs? Or, alternatively: is there a bound for the degree of polynomial first integrals? In this chapter, we show that the degrees of first integrals are related to the Kovalevskaya exponents and that a partial answer to this problem can be given.

5.1 The general approach: the variational equation

The general basis for proving the nonintegrability of a system of differential equations is the analysis of the variational equation around a particular solution. Consider the n-dimensional analytic vector field

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{C}^n, \tag{5.1}$$

together with a known particular solution $\hat{\mathbf{x}} = \hat{\mathbf{x}}(t)$. The *variational equation* around $\hat{\mathbf{x}}(t)$ is the linear system obtained by linearizing the vector field around the solution $\hat{\mathbf{x}}(t)$. It can be obtained by substituting $\mathbf{x} = \hat{\mathbf{x}}(t) + \epsilon\mathbf{u}$ and considering the system to first order in ϵ. That is,

$$\dot{\mathbf{u}} = D\mathbf{u}, \qquad \mathbf{u} \in \mathbb{C}^n, \tag{5.2}$$

where $D = D\mathbf{f}(\hat{\mathbf{x}}(t))$ is the Jacobian matrix on the particular solution. Together with the variational equation we consider the *adjoint variational equation*

$$\dot{\bar{\mathbf{u}}} = -\bar{\mathbf{u}}D, \qquad \bar{\mathbf{u}} \in \mathbb{C}^n. \tag{5.3}$$

The variables in the adjoint system are denoted by an overbar. The variational equation is a linear system of differential equations. If the nonlinear system admits some first integrals so does the variational equation. Therefore, if we can prove that the variational equation does not admit any first integral within a given class of functions, we can conclude that the original system is nonintegrable (with respect to the same class). Most theories of nonintegrability rely on this simple idea. However, depending on the choice of the particular solution, the actual analysis of the variational equation will be different. The variational equation and its adjoint have general properties independent of the choice of the particular solution.

Proposition 5.1 *There exist two fundamental solution matrices, Q and $\bar{Q}$ of the variational equation and its adjoint such that*

$$\dot{Q} = DQ, \tag{5.4.a}$$
$$\dot{\bar{Q}} = \bar{Q}D, \tag{5.4.b}$$
$$Q\bar{Q} = \bar{Q}Q = \mathrm{I}, \tag{5.4.c}$$

where $D = D\mathbf{f}(\hat{\mathbf{x}}(t))$.

Proof. Let Y, $\bar{Y}$ be two fundamental matrices of the variational equation and its adjoint. We first show that the product of two fundamental solutions $(Y\bar{Y})$ is a constant matrix. That is,

$$\begin{aligned} \frac{d}{dt}(\bar{Y}Y) &= \bar{Y}\dot{Y} + \dot{\bar{Y}}Y, \\ &= \bar{Y}DY - \bar{Y}DY, \\ &= 0. \end{aligned} \tag{5.5}$$

Since both Y and $\bar{Y}$ are fundamental matrices, so are YA and $B\bar{Y}$ for any $A, B \in \mathrm{GL}(n, \mathbb{C})$. We can choose A and B such that $Q = YA$, $\bar{Q} = \bar{B}Y$ with $YAB\bar{Y} = \mathrm{I}$. □

Proposition 5.2 *Let $I = I(\mathbf{x})$ be a first integral of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then $\bar{\mathbf{u}}(t) = \partial_{\mathbf{x}} I(\hat{\mathbf{x}}(t))$ is a solution of the adjoint.*

Proof. In order to show that $\bar{\mathbf{u}}(t) = \partial_{\mathbf{x}} I(\hat{\mathbf{x}}(t))$ is a solution of the adjoint, we compute its time derivative:

$$\begin{aligned} \dot{\bar{\mathbf{u}}} &= \frac{d}{dt}(\partial_{\mathbf{x}} I), \\ &= \mathbf{f}.\partial_{\mathbf{x}}(\partial_{\mathbf{x}} I), \\ &= \partial_{\mathbf{x}}(\mathbf{f}.\partial_{\mathbf{x}} I) - (\partial_{\mathbf{x}} I)(\partial_{\mathbf{x}}\mathbf{f}), \\ &= \partial_{\mathbf{x}}(\frac{dI}{dt}) - (\partial_{\mathbf{x}} I)D\mathbf{f}, \\ &= -\bar{\mathbf{u}}D\mathbf{f}. \end{aligned} \tag{5.6}$$

□

The following Lemma due to Poincaré is a central tool in nonintegrability theory.

Proposition 5.3 (Poincaré's lemma) *Let $I = I(\mathbf{x})$ be a first integral of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then for all solutions $\hat{\mathbf{x}}(t)$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $J = \partial_{\mathbf{x}} I(\hat{\mathbf{x}}).\mathbf{u}$ is a linear time-dependent first integral of the variational equation.*

Proof. We compute the time derivative of J. That is,

$$\begin{aligned}
\frac{dJ}{dt} &= \frac{d}{dt}(\partial_{\mathbf{x}} I.\mathbf{u}),\\
&= \mathbf{f}.(\partial_{\mathbf{x}}(\partial_{\mathbf{x}} I).\mathbf{u}) + (\partial_{\mathbf{x}} I).(D\mathbf{u}),\\
&= -(\partial_{\mathbf{x}} I).((\partial_{\mathbf{x}}, \mathbf{f})\mathbf{u}) + (\partial_{\mathbf{x}} I).(D\mathbf{u}),\\
&= -(\partial_{\mathbf{x}} I).(D\mathbf{u}) + (\partial_{\mathbf{x}} I)(D\mathbf{u})\\
&= 0.
\end{aligned} \tag{5.7}$$

□

Poincaré's lemma can also be seen as a direct consequence of the two previous propositions. Indeed, since $\partial_{\mathbf{x}} I$ is a solution of the adjoint variational equation, the relation $\bar{Q}Q =$I implies that $\partial_{\mathbf{x}} I.\mathbf{u}$ is constant. More generally, if the original vector field admits k independent first integrals and there exists $t_1 \in \mathbb{R}$ such that $\partial_{\mathbf{x}} I_1(\hat{\mathbf{x}}(t_1)), \partial_{\mathbf{x}} I_2(\hat{\mathbf{x}}(t_1)), \ldots, \partial_{\mathbf{x}} I_k(\hat{\mathbf{x}}(t_1))$ are linearly independent, then the gradients of the first integrals evaluated on $\hat{\mathbf{x}}(t)$ provides k linearly independent first integrals for the variational equation and k independent solutions for the adjoint.

5.1.1 Nonintegrability of linear systems

The simplest systems for which the existence or non-existence of first integrals can be explicitly proven are linear systems. Consider a linear system with constant coefficients

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A \in \mathrm{M}_n(\mathbb{C}), \quad \mathbf{x} \in \mathbb{C}^n. \tag{5.8}$$

Following Section 2.2, the function $I = I(\mathbf{x})$ is a formal, analytic polynomial or rational first integral if I is a formal power series, analytic polynomial or rational function and $\delta_{\mathbf{f}} I = 0$. The existence of such a first integral is guaranteed by the following results.

Theorem 5.1 *Consider a system with constant coefficients $\dot{\mathbf{x}} = A\mathbf{x}$ where A is semi-simple with eigenvalues $\boldsymbol{\lambda}$, then we have*
(1) the system admits k formal first integrals if and only if there exist k linearly independent positive integer vectors $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$ such that

$$\boldsymbol{\lambda}.\mathbf{m}^{(i)} = 0, \qquad i = 1, \ldots, k; \tag{5.9}$$

(2) the system admits k rational first integrals if and only if there exist k linearly independent integer vectors $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$ such that

$$\boldsymbol{\lambda}.\mathbf{m}^{(i)} = 0, \qquad i = 1, \ldots, k. \tag{5.10}$$

Proof. Assume without loss of generality that $A = \mathrm{diag}(\boldsymbol{\lambda})$. (1) First, assume that there exist k linearly independent positive integer vectors $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$ such that $\boldsymbol{\lambda}.\mathbf{m}^{(i)} = 0$. Then, by direct computation we verify that $I_i = \mathbf{x}^{\mathbf{m}^{(i)}}$, $i = 1, \ldots, k$ are k independent polynomial first integrals. Conversely, assume that there exist k formal power series first integrals

$$I_i = \sum_{\mathbf{n} \in \mathbb{N}^n} a_{\mathbf{n}}^{(i)} \mathbf{x}^{\mathbf{n}}, \quad i = 1, \ldots, k. \tag{5.11}$$

Since I_i is a first integral, we have $\dot{I}_i = \sum_{\mathbf{n}} a_{\mathbf{n}}^{(i)}(\boldsymbol{\lambda}.\mathbf{n})\mathbf{x}^{\mathbf{n}} = 0$ which implies that $\boldsymbol{\lambda}.\mathbf{n} = 0$ for all $\mathbf{n}$ such that $\mathbf{a}_{\mathbf{n}}^{(i)} \neq 0$. Let $\{\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \ldots, \mathbf{m}^{(l)}\}$ be the set of all such vectors. There exist l first integrals $J_i = \mathbf{x}^{\mathbf{m}^{(i)}},\ i = 1, \ldots, l$. Among the set $\{\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \ldots, \mathbf{m}^{(l)}\}$, there exist k linearly independent vectors. Assume by contradiction that there are $k' < k$ linearly independent vectors, then there are only k' independent first integrals among $J_1, \ldots, J_l$. Since the integrals $I_1, \ldots, I_k$ are functions of $J_1, \ldots, J_l$, there are only $k' < k$ independent first integrals, a contradiction. This concludes the first part of the proof.
(2) Again, assume that there exist k linearly independent integer vectors $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$ such that $\boldsymbol{\lambda}.\mathbf{m}^{(i)} = 0$. Then, we verify that $I_i = \mathbf{x}^{\mathbf{m}^{(i)}},\ i = 1, \ldots, k$ are k independent rational first integrals. Conversely, assume that there are k rational first integrals $I_i = P_i/Q_i,\ i = 1, \ldots, k$ where P and Q are relatively prime and

$$P_i = \sum_{\mathbf{p}} a_{\mathbf{p}}^{(i)} \mathbf{x}^{\mathbf{p}}, \quad Q_i = \sum_{\mathbf{q}} b_{\mathbf{q}}^{(i)} \mathbf{x}^{\mathbf{q}}. \tag{5.12}$$

Since I_i is a first integral, we have $\sum_{\mathbf{p},\mathbf{q}} a_{\mathbf{p}}^{(i)} b_{\mathbf{q}}^{(i)}((\mathbf{p}-\mathbf{q}).\boldsymbol{\lambda})\mathbf{x}^{\mathbf{p}+\mathbf{q}} = 0$ which implies that $\mathbf{m} = \mathbf{p} - \mathbf{q}$ is such that $\boldsymbol{\lambda}.\mathbf{m} = 0$ for all pairs $\mathbf{p}, \mathbf{q}$ such that $a_{\mathbf{p}}^{(i)} b_{\mathbf{q}}^{(i)} \neq 0$. Let $\{\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \ldots, \mathbf{m}^{(l)}\}$ be the set of all such vectors. Using the same argument as before, there exist $l \geq k$ independent rational first integrals $J_i = \mathbf{x}^{\mathbf{m}^{(i)}},\ i = 1, \ldots, l$ and at least k linearly independent integer valued vectors of among the set $\{\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \ldots, \mathbf{m}^{(l)}\}$. □

Definition 5.1 A finite subset $\{b_1, \ldots, b_k\}$ of a complex vector space E is $\mathbb{K}$-*independent* if $a_1 b_1 + a_2 b_2 + \ldots + a_k b_k = 0$ with $a_i \in \mathbb{K}$ implies $a_1 = a_2 = \ldots = a_k = 0$.

Corollary 5.1 (Nowicki, 1996) *Consider the system $\dot{\mathbf{x}} = A\mathbf{x}$ where the matrix A is semi-simple with eigenvalues $\boldsymbol{\lambda}$. Then, $\{\lambda_1, \ldots, \lambda_n\}$ are $\mathbb{N}$-independent if and only if there is no formal first integral. Moreover, $\{\lambda_1, \ldots, \lambda_n\}$ are $\mathbb{Z}$-independent if and only if there is no rational first integral.*

If A is not semi-simple, then the first part of the corollary is still valid. However, in the case of rational first integrals, the corollary should be replaced by the following result (see Nowicki (1996) for a proof).

Proposition 5.4 *Consider the system $\dot{\mathbf{x}} = A\mathbf{x}$ and assume it has no rational first integral. Then, either the matrix is A is semi-simple with $\mathbb{Z}$-independent eigenvalues or the Jordan form of the matrix A is (up to a permutation of rows and columns) of the form*

$$D\mathbf{f}(\mathbf{x}_*) = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \ddots & & & \vdots \\ 0 & \ldots & \ldots & 0 & \lambda_k & \ldots & 0 \\ \vdots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & \ldots & 0 & \lambda_{n-1} \end{bmatrix}, \tag{5.13}$$

and $\{\lambda_1, \lambda_2 \ldots, \lambda_{n-1}\}$ are $\mathbb{Z}$-independent.

5.2 First integrals and linear eigenvalues

Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}$ is analytic and let $\mathbf{x}_*$ be a fixed point of the vector field where the Jacobian matrix is semi-simple. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be the linear eigenvalues around the fixed points (that is, the eigenvalues of $D\mathbf{f}(\mathbf{x}_*)$). In the following, whenever we evaluate first integrals on a solution or assume the existence of multiple formal first integrals, we assume that there is a non-empty sector connected to the fixed points where all the formal first integrals and all of the formal solutions under consideration are defined. This is the case when, for instance, the first integrals are globally defined or if the first integrals are analytic around the fixed point $\mathbf{x}_*$. The following theorem states that the existence of k time-independent first integrals implies the existence of k resonances between linear eigenvalues.

Theorem 5.2 *If a system has k independent formal first integrals defined around a fixed point $\mathbf{x}_*$ with eigenvalues $\boldsymbol{\lambda}$ and a semi-simple Jacobian matrix, then there exist k linearly independent positive integer vectors $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$ such that*

$$\boldsymbol{\lambda}.\mathbf{m}^{(i)} = 0, \qquad i = 1, \ldots, k. \tag{5.14}$$

Proof. Since $D\mathbf{f}(\mathbf{x}_*)$ is semi-simple, we can assume without loss of generality that the linear part of the vector field is in diagonal form and $\mathbf{x}_* = \mathbf{0}$. Let $I_1, \ldots, I_k$ be k independent formal first integrals and let $J_1, \ldots, J_k$ be k other formal first integrals, polynomial in $I_1, \ldots, I_k$ chosen, such that

$$J_i = \mathbf{x}^{\mathbf{m}^{(i)}} + \sum_{\mathbf{n},\ |\mathbf{n}| \geq |\mathbf{m}^{(i)}|, \mathbf{n} \neq \mathbf{m}^{(i)}} a_{\mathbf{n}}^{(i)} \mathbf{x}^{\mathbf{n}}, \qquad i = 1, \ldots, k. \tag{5.15}$$

where $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$ are linearly independent positive integer vectors. Now, let $J = \mathbf{x}^{\mathbf{m}} + \sum a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$ be one of the first integrals and compute $\delta_{\mathbf{f}} J$. That is,

$$\begin{aligned} \delta_{\mathbf{f}} J &= \mathbf{f}.\partial_{\mathbf{x}} J, \\ &= (\boldsymbol{\lambda}.\mathbf{m})\mathbf{x}^{\mathbf{m}} + \sum_{\mathbf{n},\ \mathbf{n} \neq \mathbf{m}} b_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}. \end{aligned} \tag{5.16}$$

Since $\delta_{\mathbf{f}} J = 0$, we have $\boldsymbol{\lambda}.\mathbf{m} = 0$ and the result follows. □

The non-existence of polynomial, analytic or formal first integrals can be obtained as a direct consequence of the following theorem (Furta, 1996).

Corollary 5.2 *If the eigenvalues of $\mathbf{f}(\mathbf{x})$ around a fixed point $\mathbf{x}_*$, where the Jacobian matrix is semi-simple, are $\mathbb{N}$-independent, then there is no formal first integral.*

Further, if a vector field has k algebraic first integrals, a similar result can be obtained.

Theorem 5.3 *If a system has k independent algebraic first integrals $I_1, \ldots, I_k$ defined around a fixed point $\mathbf{x}_*$ with eigenvalues $\boldsymbol{\lambda}$ and semi-simple Jacobian matrix, then there exist k linearly independent integer vectors (not identically zero) $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$ such that*

$$\boldsymbol{\lambda}.\mathbf{m}^{(i)} = 0, \qquad i = 1, \ldots, k. \tag{5.17}$$

Proof. Again, since $D\mathbf{f}(\mathbf{x}_*)$ is semi-simple, we can assume without loss of generality that the linear part of the vector field is in diagonal form and $\mathbf{x}_* = \mathbf{0}$. By Brun's theorem (Theorem 2.4), the existence of k algebraic first integrals implies the existence of k independent rational first integrals $J_1 = P_1/Q_1, \ldots, J_k = P_k/Q_k$, chosen such that

$$P_i = \alpha_i \mathbf{x}^{\mathbf{p}^{(i)}} + \sum_{\mathbf{n},\ |\mathbf{n}| \geq |\mathbf{p}^{(i)}|, \mathbf{n} \neq \mathbf{p}^{(i)}} a_{\mathbf{n}}^{(i)} \mathbf{x}^{\mathbf{n}}, \qquad i = 1, \ldots, k, \tag{5.18.a}$$

$$Q_i = \beta_i \mathbf{x}^{\mathbf{q}^{(i)}} + \sum_{\mathbf{n},\ |\mathbf{n}| \geq |\mathbf{q}^{(i)}|, \mathbf{n} \neq \mathbf{q}^{(i)}} b_{\mathbf{n}}^{(i)} \mathbf{x}^{\mathbf{n}}, \qquad i = 1, \ldots, k, \tag{5.18.b}$$

with

$$P_i Q_i = \alpha_i \beta_i \mathbf{x}^{\mathbf{p}^{(i)} + \mathbf{q}^{(i)}} + \sum_{\mathbf{n},\ |\mathbf{n}| \geq |\mathbf{p}^{(i)}| + |\mathbf{q}^{(i)}|, \mathbf{n} \neq \mathbf{p}^{(i)} + \mathbf{q}^{(i)}} c_{\mathbf{n}}^{(i)} \mathbf{x}^{\mathbf{n}}, \tag{5.19}$$

and where $\{\mathbf{m}^{(1)} = \mathbf{p}^{(1)} - \mathbf{q}^{(1)}, \ldots, \mathbf{m}^{(k)} = \mathbf{p}^{(k)} - \mathbf{q}^{(k)}\}$ are linearly independent integer vectors. Let $J = P/Q$ be one of the first integrals and compute $\delta_{\mathbf{f}} J = 0$, that is, $Q\delta_{\mathbf{f}} P - P\delta_{\mathbf{f}} Q = 0$:

$$\begin{aligned} 0 &= Q\delta_{\mathbf{f}} P - P\delta_{\mathbf{f}} Q \\ &= \alpha\beta(\boldsymbol{\lambda}.(\mathbf{p} - \mathbf{q}))\mathbf{x}^{\mathbf{p}+\mathbf{q}} + \sum_{\mathbf{n},\ \mathbf{n} \neq \mathbf{m}} d_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}. \end{aligned} \tag{5.20}$$

We conclude that $\boldsymbol{\lambda}.\mathbf{m} = 0$ with $\mathbf{m}$ a vector of integers. □

Corollary 5.3 *If the linear eigenvalues of $\mathbf{f}(\mathbf{x})$ around a fixed point $\mathbf{x}_*$, where the Jacobian matrix is semi-simple, are $\mathbb{Z}$-independent, then there is no rational (hence algebraic) first integral.*

If the Jacobian matrix at the fixed point is not semi-simple, then Corollary 5.2 is still valid. However, in that case, the set of eigenvalues is always $\mathbb{Z}$-dependent (since at least two eigenvalues are identical) and Corollary 5.3 should be replaced by the following result.

Proposition 5.5 *If the Jordan form of the Jacobian matrix of $\mathbf{f}(\mathbf{x})$ at a fixed point $\mathbf{x}_*$ is of the form (up to permutation of rows and columns)*

$$D\mathbf{f}(\mathbf{x}_*) = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & & & \vdots \\ 0 & \dots & \dots & 0 & \lambda_k & \dots & 0 \\ \vdots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \dots & 0 & \lambda_{n-1} \end{bmatrix}, \tag{5.21}$$

and $\{\lambda_1, \lambda_2 \dots, \lambda_{n-1}\}$ are $\mathbb{Z}$-independent, then there is no rational (hence algebraic) first integral.

Now, if the system admits a time-independent first integral, we have the following result (the proof is left as an exercise).

Proposition 5.6 *If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ admits a time-dependent first integral of the form $I = Pe^{-\chi t}$ where P is polynomial (resp. rational) in $\mathbf{x}$, then there exists a vector $\mathbf{m}$ of positive integer (resp. integer) such that*

$$\boldsymbol{\lambda}.\mathbf{m} = \chi, \tag{5.22}$$

where $\boldsymbol{\lambda}$ are the linear eigenvalues around a fixed point $\mathbf{x}_*$.

Example 5.1 **Bianchi IX model.** The system

$$\dot{x}_1 = (1 + x_4)(1 + x_1x_2 + x_3(x_1 + x_2) - x_4^2) - 2x_1x_3, \tag{5.23.a}$$
$$\dot{x}_2 = (1 - x_4)(1 + x_1x_2 + x_3(x_1 + x_2) - x_4^2) - 2x_1x_3, \tag{5.23.b}$$
$$\dot{x}_3 = 2(x_4^2 - x_3^2 - 1), \tag{5.23.c}$$
$$\dot{x}_1 = x_4(x_1 + x_2 - 2x_3) + \frac{1}{2}(x_1 - x_2)(1 - x_1x_2 - x_3(x_1 + x_2) + x_4^2), \tag{5.23.d}$$

is a reduction of the Bianchi B(IX) cosmological model (Maciejewski & Szydlowski, 1998). Around the fixed point $\mathbf{x}_* = (-i, -i, i, 0)$, the system has the linear eigenvalues $\boldsymbol{\lambda} = (-2i, -2i, -4i, -4i)$. Therefore, we conclude that around $\mathbf{x}_*$ there is no formal first integral. In particular, this implies the non-existence of polynomial first integrals. ■

5.3 First integrals and Kovalevskaya exponents

In the previous section, we studied the variational equations around the fixed points and related the existence of first integrals to the existence of resonance relations between linear eigenvalues. We now consider another particular solution, the scale-invariant solutions $\hat{\mathbf{x}} = \boldsymbol{\alpha} t^{\mathbf{p}}$, and relate the weighted degrees of first integrals with the Kovalevskaya exponents.

5.3.1 Yoshida's analysis

One of the pioneering works in the theory of nonintegrability is due to Yoshida (1983a; 1983b). Using singularity analysis, he was able to derive necessary conditions for algebraic integrability based on Kovalevskaya exponents. Yoshida's analysis was initially performed on scale-invariant systems. However, as we show here, it is valid for general systems. Indeed, if a vector field admits a first integral, then its weight-homogeneous component of highest degree also admits a first integral (Proposition 2.3). Therefore, to test for the non-existence of first integrals, we restrict the analysis to the component of highest degree of all weight-homogeneous decomposition. These components are scale-invariant systems.

Consider a scale-invariant system, (with respect to a weight $\mathbf{w} = -\mathbf{p}$), $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and assume that there exists a scale-invariant solution of the form

$$\mathbf{x} = \boldsymbol{\alpha} t^{\mathbf{p}}, \tag{5.24}$$

where the coefficients $\boldsymbol{\alpha} \in \mathbb{C}^n$ are given by one of the non vanishing solutions of the the algebraic equation

$$\mathbf{p}\boldsymbol{\alpha} = \mathbf{f}(\boldsymbol{\alpha}). \tag{5.25}$$

For a given $\mathbf{p}$, there may exist different sets of values $\boldsymbol{\alpha}$, referred to as different *balances* in agreement with the definition of Chapter 3. We consider one of these solutions and the Kovalevskaya matrix

$$K = D\mathbf{f}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p}), \tag{5.26}$$

where $D\mathbf{f}(\boldsymbol{\alpha})$ is the Jacobian matrix evaluated on $\mathbf{x} = \boldsymbol{\alpha}$. Here again, the eigenvectors of K can be used to build particular solutions to the variational equations. The Kovalevskaya exponents $\boldsymbol{\rho} = \{\rho_1 = -1, \rho_2, \dots, \rho_n\}$ are the eigenvalues of K.

Yoshida's results are twofold. First he proves that under certain conditions, the weighted degree of a first integral is a Kovalevskaya exponent. Second, he shows that if one of the Kovalevskaya exponents is not rational, then the system cannot be algebraically integrable.

Theorem 5.4 (Yoshida) *Let $I(\mathbf{x})$ be a weight-homogeneous first integral of weighted degree d for the scale-invariant system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. If $\partial_{\mathbf{x}} I(\boldsymbol{\alpha})$ is not identically zero then d is a Kovalevskaya exponent.*

Proof. The scale-invariant system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ admits the particular solution $\mathbf{x} = \boldsymbol{\alpha} t^{\mathbf{p}}$. We can therefore apply Poincaré's lemma (Proposition 5.3). The existence of $I(\mathbf{x})$ implies the existence of a first integral $J = \partial_{\mathbf{x}} I(\boldsymbol{\alpha} t^{\mathbf{p}})\mathbf{u}$ for the variational equation

$$\dot{\mathbf{u}} = D\mathbf{f}(\boldsymbol{\alpha} t^{\mathbf{p}})\mathbf{u}. \tag{5.27}$$

The general solution of the variational equation is $\mathbf{u} = \sum_{i=1}^{k} \boldsymbol{\beta}^{(i)} t^{\rho_i}$ where $\boldsymbol{\beta}^{(i)}$ is polynomial in $\log t$ and can be expressed in terms of the generalized eigenvectors of $K = D\mathbf{f}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p})$. This solution contains n arbitrary parameters. We can therefore evaluate the first integral J on this solution to obtain

$$\begin{aligned} J &= \sum_{i=1}^{k} \partial_{\mathbf{x}}(\boldsymbol{\alpha}) t^{-\mathbf{p}-d} \boldsymbol{\beta}^{(i)} t^{\mathbf{p}+\rho_i}, \\ &= \sum_{i=1}^{k} t^{-d+\rho_i} \partial_{\mathbf{x}}(\boldsymbol{\alpha}) \boldsymbol{\beta}^{(i)}. \end{aligned} \tag{5.28}$$

If K is semi-simple then $k = n$ and the eigenvectors $\boldsymbol{\beta}^{(i)}$ form a set of n linearly independent vectors and at least one of them, say $i = j$, is such that $\partial_{\mathbf{x}}(\boldsymbol{\alpha})\boldsymbol{\beta}^{(j)} \neq 0$. Since J is constant in time, we have $d = \rho_j$. If K is not semi-simple, then there is a complete set of generalized eigenvectors $\boldsymbol{\gamma}^{(1)}, \dots, \boldsymbol{\gamma}^{(n)}$ for K and for at least one of them $\partial_{\mathbf{x}}(\boldsymbol{\alpha})\boldsymbol{\gamma}^{(i)} \neq 0$ and the result follows. □

Example 5.2 A Hamiltonian system. As an example, consider the following three-degree-of-freedom Hamiltonian system (Hietarinta, 1987)

$$H = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2\right) + \left(x_1^4 + 16x_2^4 + \mu x_3^4 + 12x_1^2x_2^2\right). \tag{5.29}$$

This Hamiltonian system is weight-homogeneous with respect to the weight vector $\mathbf{w} = (1, 1, 1, 2, 2, 2)$ (where $\mathbf{x} = (x_1, x_2, x_3, y_1, y_2, y_3)$). Moreover, it is integrable with second and third first integrals given by

$$I_1 = y_3^2 + 2\mu x_3^4, \tag{5.30.a}$$

$$I_2 = x_2y_1^2 - x_1y_1y_2 - 8x_1^2x_2^3 - 4x_1^3x_2^2. \tag{5.30.b}$$

The degrees of H, I_1, and I_2 with respect to $\mathbf{w}$ are, respectively, $d_h = 4$, $d_1 = 4$, and $d_2 = 5$. The first step of Yoshida's analysis consists of finding all possible dominant balances, that is, all scale-invariant solutions $\mathbf{x} = \boldsymbol{\alpha} t^{\mathbf{p}}$ with $\mathbf{p} = -\mathbf{w}$. There are 24 different solutions for $\boldsymbol{\alpha}$. For each dominant balance, we compute both the Kovalevskaya exponents $\boldsymbol{\rho}$ using relation (5.26), and the gradients of the first integrals estimated on the scale-invariant solutions. We illustrate Yoshida's theorem by considering three different dominant balances:

$$\begin{aligned} &\boldsymbol{\alpha}^{(1)} = (\frac{i}{\sqrt{2}}, 0, \frac{i}{\sqrt{2\mu}}, \frac{-i}{\sqrt{2}}, 0, \frac{-i}{\sqrt{2\mu}}), \quad \boldsymbol{\rho}^{(1)} = \{-2, -1, -1, 4, 4, 5\} \\ &\partial_{\mathbf{x}} H(\boldsymbol{\alpha}^{(1)}) \neq \mathbf{0},\ \partial_{\mathbf{x}} I_1(\boldsymbol{\alpha}^{(1)}) \neq \mathbf{0},\ \partial_{\mathbf{x}} I_2(\boldsymbol{\alpha}^{(1)}) \neq \mathbf{0}, \end{aligned} \tag{5.31.a}$$

$$\begin{aligned} &\boldsymbol{\alpha}^{(2)} = (\frac{1}{2}, \frac{i}{2\sqrt{2}}, \frac{i}{\sqrt{2\mu}}, -\frac{1}{2}, \frac{-i}{2\sqrt{2}}, \frac{-i}{\sqrt{2\mu}}), \quad \boldsymbol{\rho}^{(2)} = \{-2, -1, -1, 2, 4, 5\}, \\ &\partial_{\mathbf{x}} H(\boldsymbol{\alpha}^{(2)}) \neq \mathbf{0},\ \partial_{\mathbf{x}} I_1(\boldsymbol{\alpha}^{(2)}) = \mathbf{0},\ \partial_{\mathbf{x}} I_2(\boldsymbol{\alpha}^{(2)}) \neq \mathbf{0}, \end{aligned} \tag{5.31.b}$$

$$\begin{aligned} &\boldsymbol{\alpha}^{(3)} = (-\frac{1}{2}, \frac{-i}{2\sqrt{2}}, 0, -\frac{1}{2}, \frac{i}{2\sqrt{2}}, 0), \quad \boldsymbol{\rho}^{(3)} = \{-2, -1, 1, 2, 4, 5\}, \\ &\partial_{\mathbf{x}} H(\boldsymbol{\alpha}^{(3)}) \neq \mathbf{0},\ \partial_{\mathbf{x}} I_1(\boldsymbol{\alpha}^{(3)}) = \mathbf{0},\ \partial_{\mathbf{x}} I_2(\boldsymbol{\alpha}^{(3)}) = \mathbf{0}. \end{aligned} \tag{5.31.c}$$

In the first case, the gradients of the first integrals do not vanish on the scale-invariant solution. As a consequence, d_h, d_1 and d_2 are Kovalevskaya's exponents. In the second case, $\partial_{\mathbf{x}} I_1$ vanishes identically and d_1 is not a Kovalevskaya exponent while in the third case $\partial_{\mathbf{x}} H(\boldsymbol{\alpha})$ does not vanish and accordingly, d_h is a Kovalevskaya exponent. ■

Although, this result is a first bridge between the degrees of first integrals and singularity analysis, it is not of great predictive power. While the Kovalevskaya exponents can be computed in a finite number of steps, the functional form of the first integral is not known a priori. Therefore, the first integrals may not satisfy the assumptions. In particular, it does not forbid the existence of a first integral of higher degree for which $\partial_{\mathbf{x}} I(\boldsymbol{\alpha})$ could vanish identically. The converse statement is also true. It will be proved in the next section that $\partial_{\mathbf{x}} I(\boldsymbol{\alpha}) \neq \mathbf{0}$ if and only if d (the degree of I with respect to to $\mathbf{p}$) is a Kovalevskaya exponent for the balance under consideration (where d is considered here with the proper algebraic multiplicity). Interestingly, this result seems to be valid outside the class of algebraic first integrals since Yoshida's argument does not rely on the fact that I is an algebraic function, but only on the weight-homogeneity of the vector-field and first integral.

For Hamiltonian systems, there is an interesting relation between the Kovalevskaya exponents which was first pointed out by Yoshida and given in its final form by Lochak (1985).

Proposition 5.7 (Lochak, 1985) *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a system whose Hamiltonian is H. If ρ is a Kovalevskaya exponent for the system then so is $h - 1 - \rho$ (where h is the weighted degree of the Hamiltonian H).*

That is, for Hamiltonian systems, the Kovalevskaya exponents always come by pairs. This is analogous to the linearized eigenvalues at a fixed point (this analogy is more than formal and can be made rigorous–see Exercises). The next statement is the main result of Yoshida; it connects the occurrence of irrational Kovalevskaya exponents with algebraic nonintegrability. A system is *algebraically integrable in the weak sense* if there exist $1 \le k \le n-1$ independent algebraic first integrals $I_i(\mathbf{x}) = K_i$ $(i = 1, \dots, k)$. These k first integrals define an $(n-k)$-dimensional algebraic variety. In addition, there must exist other $(n-1-k)$ independent first integrals given by the integral of a total differential defined on an algebraic variety (see Definition 2.18).

> *If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is algebraically integrable in the weak sense, then all Kovalevskaya exponents are rational.*

However, despite the fact that this result has been widely applied and frequently verified, this last statement is not correct as illustrated on the next example due to Kummer *et al.* (1991)

Example 5.3 Counterexample. Consider the Liouville integrable Hamiltonian

$$H = p_1(p_1^2 + x_1^2) + x_1(p_2^2 + x_2^2), \tag{5.32}$$

with second first integral $I = p_2^2 + x_2^2$. This system is homogeneous with weight vector $\mathbf{w} = -\mathbf{p} = (1,1,1,1)$ and has three different balances $\boldsymbol{\alpha}^{(1)} = (-1,0,0,0)$ and $\boldsymbol{\alpha}^{(2,3)} = (1/2, 0, \pm i/2, 0)$. The corresponding Kovalevskaya exponents are $\boldsymbol{\rho}^{(1)} = \{-1, 3, 1 \pm 2i\}$ and $\boldsymbol{\rho}^{(2,3)} = \{-1, 3, 1 \pm i\}$. Therefore, the system is Liouville integrable despite the occurrence of complex Kovalevskaya exponents. ■

The correct statement of Yoshida's theorem is only related to the stronger definition of algebraic integrability (see Corollary 5.4).

> *If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is algebraically integrable, then all Kovalevskaya exponents are rational.*

5.3.2 Resonances between Kovalevskaya exponents

Yoshida's result only concerns complete integrability. However, following the same arguments, his result can be generalized to study partial integrability or nonintegrability. In this section, we establish two fundamental equalities between the Kovalevskaya exponents for non-homogeneous systems and the degrees of first integrals (Goriely, 1996).

Theorem 5.5 *If a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has k independent formal first integrals $I_1, \dots, I_k$ and a dominant balance $\hat{\mathbf{x}} = \boldsymbol{\alpha} t^{\mathbf{p}}$ with Kovalevskaya exponents $\boldsymbol{\rho} = (-1, \mathbf{r})$ $(\mathbf{r} \in \mathbb{C}^{n-1})$ and a semi-simple Kovalevskaya matrix, then there exist k linearly independent (over $\mathbb{N}$) positive integer vectors $\{\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(k)}\} \subset \mathbb{N}^{n-1}$ such that*

$$\mathbf{r}.\mathbf{m}^{(i)} = d_i \in \mathbb{Q}, \qquad i = 1, \dots, k. \tag{5.33}$$

Proof. Let $\mathbf{f}^{(0)}$ be the component of highest weighted degree associated with the homogeneous decomposition given by the dominant balance $\hat{\mathbf{x}} = \boldsymbol{\alpha} t^{\mathbf{p}}$. For the weight-homogeneous system $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$, the existence of k independent formal first integrals $I_1, \dots, I_k$ implies the existence of k independent weight-homogeneous polynomial first integrals $J_1, \dots, J_k$. Let $d_i = -\deg(J_i, \mathbf{p})$, $i = 1, \dots, k$ be the weighted degrees of $J_1, \dots, J_k$ with respect to $\mathbf{p}$. Now, consider the balance given by the solution $\hat{\mathbf{x}} = \boldsymbol{\alpha} t^{\mathbf{p}}$. Following Section 3.8.5, the companion system of $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ is obtained by introducing the variables $\mathbf{X} = t^{-\mathbf{p}}\mathbf{x}$, $X_N = e^{qs}$ and $t = e^s$. This companion system has a fixed point located at $\mathbf{X}_* = (\boldsymbol{\alpha}, 0)$. Therefore, we apply a linear transformation $\mathbf{X} - \mathbf{X}_* = M\mathbf{Y}$ such that, in the variables $\mathbf{Y}$, the companion system is $\mathbf{Y}' = \mathbf{F}(\mathbf{Y}) = \mathrm{diag}(-1, \mathbf{r}, q)\mathbf{Y} + \mathbf{G}(\mathbf{Y})$ where $\mathbf{G}(\mathbf{Y})$ does not contain any linear term. The fixed point $\mathbf{X} = \mathbf{X}_*$ is now $\mathbf{Y} = 0$ and $Y_N = X_N$. Let $\hat{\mathbf{Y}} = (Y_2, \dots, Y_{N-1})$ and $Z = Y_N$. Consider the first integrals $J_1, \dots, J_k$ written in terms of the variables $\mathbf{Y}$,

$$\begin{aligned} J_i(\mathbf{Y}) &= J_i(t^{\mathbf{p}}(\mathbf{X}_* + M\mathbf{Y})), \\ &= Z^{d_i} \sum_m \left(\sum_{0 < \mathbf{p}.\mathbf{n} \le d_i} a_{\mathbf{n}}^{(i)} Y_1^m \hat{\mathbf{Y}}^{\mathbf{n}} \right), \quad i = 1, \dots, k. \end{aligned} \tag{5.34}$$

Since J_i is a first integral, we have $\partial_{Y_1} J_i = 0$. Indeed, the first column vector of M is the eigenvector of eigenvalue $\rho = -1$. According to Proposition 3.3, this eigenvector is proportional to $\mathbf{f}^{(0)}(X_*)$. Therefore, $\partial_{Y_1} J_i$

is a sum of $\partial_{\mathbf{x}} J_i(\mathbf{X}_*)\mathbf{f}^{(0)}(\mathbf{X}_*)$ and higher derivatives which all vanish identically. As before, we can choose $\hat{J}_1, \dots, \hat{J}_k$ polynomial in $J_1, \dots, J_k$ such that

$$\hat{J}_i = Z^{d_i} \left(\hat{\mathbf{Y}}^{\mathbf{m}^{(i)}} + \sum_{\mathbf{n}, |\mathbf{n}| \geq |\mathbf{m}^{(i)}|} a_{\mathbf{n}}^{(i)} \hat{\mathbf{Y}}^{\mathbf{n}} \right), \quad i = 1, \dots, k, \tag{5.35}$$

where $\{\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(k)}\}$ are linearly independent positive integer vectors. The condition $\delta_{\mathbf{f}^{(0)}} J_i = 0$ implies $\delta_{\mathbf{F}} \hat{J}_i = 0$, that is, $\mathbf{r}.\mathbf{m}^{(i)} = d_i$ for $i = 1, \dots, k$. □

Theorem 5.6 *If a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has k independent algebraic first integrals $I_1, \dots, I_k$ and a dominant balance $\hat{\mathbf{x}} = \boldsymbol{\alpha} t^{\mathbf{p}}$ with Kovalevskaya exponents $\boldsymbol{\rho} = (-1, \mathbf{r})$ and a semi-simple Kovalevskaya matrix, then there exist k linearly independent integer vectors $\{\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(k)}\} \subset \mathbb{Z}^{n-1}$ such that*

$$\mathbf{r}.\mathbf{m}^{(i)} = d_i, \quad i = 1, \dots, k. \tag{5.36}$$

Proof. We follow the steps of the previous proof and consider the system $\mathbf{Y}' = \mathbf{F}(\mathbf{Y}) = \text{diag}(-1, \mathbf{r}, 1)\mathbf{Y} + \mathbf{G}(\mathbf{Y})$ around $\mathbf{Y} = 0$. The existence of k algebraic first integrals for $\delta_{\mathbf{f}}$ implies the existence of k rational weight-homogeneous first integrals for $\delta_{\mathbf{f}^{(0)}}$ of weighted degree $d_i = -\deg(J_i, \mathbf{p})$. In turn, this implies the existence of k rational first integrals for $\delta_{\mathbf{F}}$, $J_1 = P_1/Q_1, \dots, J_k = P_k/Q_k$ which are functions of $\hat{\mathbf{Y}} = (Y_2, \dots, Y_n)$ and are chosen such that

$$P_i = Z^{\mathbf{p}.\mathbf{s}^{(i)}} \left(\alpha_i \hat{\mathbf{Y}}^{\mathbf{s}^{(i)}} + \sum_{\mathbf{n}.\mathbf{p} \geq (\mathbf{s}^{(i)}, \mathbf{p}), \mathbf{n} \neq \mathbf{s}^{(i)}} a_{\mathbf{n}}^{(i)} \hat{\mathbf{Y}}^{\mathbf{n}} \right), \tag{5.37.a}$$

$$Q_i = Z^{\mathbf{p}.\mathbf{q}^{(i)}} \left(\beta_i \hat{\mathbf{Y}}^{\mathbf{q}^{(i)}} + \sum_{\mathbf{n}.\mathbf{p} \geq (\mathbf{q}^{(i)}, \mathbf{p}), \mathbf{n} \neq \mathbf{p}^{(i)}} b_{\mathbf{n}}^{(i)} \hat{\mathbf{Y}}^{\mathbf{n}} \right), \tag{5.37.b}$$

with

$$P_i Q_i = Z^{-d_i} \left(\alpha_i \beta_i \hat{\mathbf{Y}}^{\mathbf{s}^{(i)} + \mathbf{q}^{(i)}} + \sum_{\mathbf{n}.\mathbf{p} \geq (\mathbf{s}^{(i)} + \mathbf{q}^{(i)}, \mathbf{p}), \mathbf{n} \neq \mathbf{s}^{(i)} + \mathbf{q}^{(i)}} c_{\mathbf{n}}^{(i)} \hat{\mathbf{Y}}^{\mathbf{n}} \right),$$

and where $\{\mathbf{m}^{(1)} = \mathbf{s}^{(1)} - \mathbf{q}^{(1)}, \dots, \mathbf{m}^{(k)} = \mathbf{s}^{(k)} - \mathbf{q}^{(k)}\}$ are linearly independent integer vectors. Let $J = P/Q$ be one of the first integrals. If we compute $\delta_{\mathbf{F}} J = 0$, that is,

$$\begin{aligned} 0 &= Q\delta_{\mathbf{F}} P - P\delta_{\mathbf{F}} Q, \\ &= Z^{-d} \left(\alpha\beta(-d + \mathbf{r}.(\mathbf{s} - \mathbf{q}))\hat{\mathbf{Y}}^{\mathbf{p}+\mathbf{q}} + \sum_{\mathbf{n},\ \mathbf{n} \neq \mathbf{m}} d_{\mathbf{n}} \hat{\mathbf{Y}}^{\mathbf{n}} \right), \end{aligned} \tag{5.38}$$

we conclude that $\mathbf{r}.\mathbf{m} = d$. □

As a corollary, we obtain Yoshida's theorem (1983b).

Corollary 5.4 *If there is at least one irrational or imaginary Kovalevskaya exponent, the system is not algebraically integrable.*

Proof. From the previous proposition, we know that if the system has $(n-1)$ first integrals, then there are $(n-1)$ linearly independent relations $\mathbf{r}.\mathbf{m}^{(i)} = d_i,\ i = 1, \dots, n-1$. Let $\mathbf{d} = (\mathbf{p}.\mathbf{m}^{(1)}, \dots, \mathbf{p}.\mathbf{m}^{(n-1)})$. Therefore, there exists a matrix $N \in \text{GL}(n-1, \mathbb{Z})$, such that $N\mathbf{r} = \mathbf{d}$. We find $\mathbf{r} = N^{-1}\mathbf{d}$ which implies $\mathbf{r} \in \mathbb{Q}^{n-1}$. □

Yoshida's theorem gives a necessary condition for complete integrability. Conversely, we now find sufficient conditions for complete nonintegrability, that is, the non-existence of at least one first integral (This result was proved in different settings (Yoshida, 1983b; Sadetov, 1993; Moulin-Ollagnier *et al.*, 1995; Goriely, 1996; Furta, 1996; Emelyanov & Tsygvintsev, 2000) and generalized to tensor equations by Kozlov (1992)).

Corollary 5.5 *If there is a dominant balance such that the Kovalevskaya matrix is semi-simple and the Kovalevskaya exponents* $\rho_1, \ldots, \rho_n$ *are* $\mathbb{Z}$*-independent, then there is no rational first integral.*

Proof. Assume, by contradiction that there is a rational first integral. Then, there exist integers $i_2, \ldots, i_n$ such that $i_2\rho_2 + \ldots + i_n\rho_n = d$ where $d \in \mathbb{Q}$. This relation can be written $j_1\rho_1 + \ldots + j_n\rho_n = 0$ for some integers $j_1, \ldots, j_n$. However, this is impossible since $\rho_1, \ldots, \rho_n$ are $\mathbb{Z}$-independent. □

Corollary 5.6 *If there is a dominant balance such that the Kovalevskaya matrix is semi-simple and the Kovalevskaya exponents* $\rho_1, \ldots, \rho_n$ *are* $\mathbb{N}$*-independent, then there is is no polynomial first integral.*

Proof. If the Kovalevskaya matrix is semi-simple and there exists a polynomial first integral, the fundamental relation between the Kovalevskaya exponents and the degrees of first integrals reads $i_2\rho_2 + \ldots + i_n\rho_n = d$, with $i_j \in \mathbb{N}\ \forall\ j$ and the results follows as before. □

If the Kovalevskaya matrix is not semi-simple then the last corollary is still true and can be proved by using the main theorem of Nowicki (1996) applied on the linear part of the companion system. However, in the case of a rational first integral, we must modify the result to take into account a Jordan block of the type given in Proposition 5.4. If a single first integral $I = I(\mathbf{x})$ is known then the last results cannot be applied since the Kovalevskaya exponents are rationally related. However, if $\partial_{\mathbf{x}} I(\boldsymbol{\alpha}) \neq 0$, then we can exclude the corresponding exponent from the discussion and we have the following result (the proof is left as an exercise (Furta, 1996)).

Proposition 5.8 *Assume that the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *has a first integral* $I = I(\mathbf{x})$ *such that for a balance* $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$, $\partial_{\mathbf{x}} I(\boldsymbol{\alpha}) \neq 0$. *Let* $d = \deg(I, \mathbf{p})$ *and* $\mathcal{R} = \{\rho_1, \ldots, \rho_n\} \setminus \{d\}$. *If the set* $\mathcal{R}$ *is* $\mathbb{Z}$*-independent then there is no rational first integral.*

Example 5.4 The Jouanolou system. One of the first proofs of the algebraic nonintegrability of a dynamical system was given by Jouanolou (1979). The system in question reads

$$\dot{x}_1 = x_3^n, \tag{5.39.a}$$
$$\dot{x}_2 = x_1^n, \tag{5.39.b}$$
$$\dot{x}_3 = x_2^n, \tag{5.39.c}$$

where $n \geq 2$ is a natural number. The system is homogeneous and all dominant balances $\hat{\mathbf{x}} = \boldsymbol{\alpha}t^{\mathbf{p}}$ have $\mathbf{p} = (-1/(n-1), -1/(n-1), -1/(n-1))$. The Kovalevskaya matrix is semi-simple and the Kovalevskaya exponents are $\boldsymbol{\rho} = (-1, \frac{(1\pm i\sqrt{3})n+1}{2(n-1)})$. Therefore, using Corollary 5.4, we can conclude that the system does not admit a pair of rational first integrals. However, the analysis of the Kovalevskaya exponents is not enough to guarantee the non-existence of a single rational first integral and global properties have to be taken into account. Moulin-Ollagnier *et al.* (1995) used Bezout's theorem to prove that the system does not admit any Darboux polynomials. Hence, it does not have a rational or algebraic first integral. ■

Example 5.5 The Halphen system. The quadratic system

$$\dot{x}_1 = x_2x_3 - x_1(x_2 + x_3), \tag{5.40.a}$$
$$\dot{x}_2 = x_3x_1 - x_2(x_3 + x_1), \tag{5.40.b}$$
$$\dot{x}_3 = x_1x_2 - x_3(x_1 + x_2), \tag{5.40.c}$$

is the Darboux-Brioschi-Halphen system (Halphen, 1921; Maciejewski & Strelcyn, 1995). This system is homogeneous and there is a unique dominant balance $\hat{\mathbf{x}} = \boldsymbol{\alpha}t^{\mathbf{p}}$ with $\mathbf{p} = (-1, -1, -1)$ and $\boldsymbol{\alpha} = (1, 1, 1)$. The Kovalevskaya matrix is semi-simple and the Kovalevskaya exponents are $\boldsymbol{\rho} = (-1, -1, -1)$. Therefore, we conclude that the Halphen system does not admit a polynomial (hence formal) first integral.

Also, it can be proved that the Halphen system does not admit a rational first integral (Maciejewski & Strelcyn, 1995). To do so, the authors combine the relations provided by Theorem 5.7 with a global property of the system (in this case, a global symmetry of the system). ■

Example 5.6 The Oregonator. The perturbed Oregonator model is a simplified model for the Belousov-Zhabotinsky chemical reaction (Tyson, 1978; Furta, 1996). It reads

$$\dot{x}_1 = a_1 x_2 - a_1 x_1(1 - a_2 x_1 + x_2 - \epsilon x_3), \tag{5.41.a}$$
$$\dot{x}_2 = a_1^{-1}(-x_2 - x_1 x_2 + a_3 x_3), \tag{5.41.b}$$
$$\dot{x}_3 = a_4(x_1 - x_3 - \epsilon x_1 x_3), \tag{5.41.c}$$

where $a_1, a_2, a_3, a_4, \epsilon$ are parameters. There are two dominant balances $\hat{\mathbf{x}} = \boldsymbol{\alpha} t^{\mathbf{p}}$ with $\mathbf{p} = (-1, -1, 0)$ and $\boldsymbol{\alpha}^{(1)} = (a_1, a_1^{-1} - a_1 a_2, \epsilon^{-1})$ or $\boldsymbol{\alpha}^{(2)} = (a_1^{-1} a_2^{-1}, 0, \epsilon^{-1})$. The corresponding weight-homogeneous component, $\mathbf{f}^{(0)}$, is

$$\dot{x}_1 = -a_1 x_1(a_2 x_1 + x_2), \tag{5.42.a}$$
$$\dot{x}_2 = -a_1^{-1} x_1 x_2, \tag{5.42.b}$$
$$\dot{x}_3 = a_4 x_1(1 - \epsilon x_3). \tag{5.42.c}$$

The Kovalevskaya exponents are $\boldsymbol{\rho}^{(1)} = (-1, -a_4 a_1 \epsilon, 1 - {a_1}^2 a_2)$ and $\boldsymbol{\rho}^{(2)} = (-1, 1 - \frac{1}{a_1^2 a_2}, -\frac{a_4 \epsilon}{a_1 a_2})$. The relation $m_2\rho_2 + m_3\rho_3 \in \mathbb{Q}$ implies that for given values of a_1, a_2, a_3, and a_4 there exist open sets of arbitrarily small values for ϵ such that the system is not algebraically integrable, ■

Example 5.7 The Lotka-Volterra system. Consider an n-dimensional Lotka-Volterra system (Moulin-Ollagnier *et al.*, 1995),

$$\dot{x}_i = \sum_{j=1}^{n} L_{ij} x_j + x_i \sum_{j=1}^{n} M_{ij} x_j, \qquad i = 1, \ldots, n, \tag{5.43}$$

where $M, L \in \mathrm{M}_n(\mathbb{C})$.

Proposition 5.9 *If any column vector of M is $\mathbb{N}$-independent (resp. $\mathbb{Z}$-independent) then there is no formal (resp. algebraic) first integral.*

Proof. To compute the Kovalevskaya exponents, we only consider the homogeneous system $\dot{\mathbf{x}} = \mathbf{x}(M\mathbf{x})$. Assume that one of the columns of M is $\mathbb{K}$-independent (where $\mathbb{K}$ is $\mathbb{Z}$ or $\mathbb{N}$) and that without loss of generality this is the first column. Therefore $M_{11} \neq 0$ and the system has a dominant balance $\mathbf{x} = \boldsymbol{\alpha} t^{-\mathbf{1}}$ with $\boldsymbol{\alpha} = (-M_{11}^{-1}, \mathbf{0})$. The corresponding Kovalevskaya exponents are $\boldsymbol{\rho} = (-1, 1 - M_{11}^{-1} M_{21}, 1 - M_{11}^{-1} M_{31}, \ldots, 1 - M_{11}^{-1} M_{n1})$. By contradiction, assume that there exists a polynomial (resp. rational) first integral. Then, there exists a vector $\mathbf{m}$ of positive integers (resp. integers) such that $m_2\rho_2 + \ldots + m_n\rho_n = d$ where $d \leq |\mathbf{m}|$. This in turn implies that there exists a vector $\mathbf{n}$ of positive integers (resp. integers) such that $n_1 M_{11} + n_2 M_{21} + \ldots + n_n M_{n1} = 0$. However, this is impossible since the the entries $M_{11}, \ldots, M_{n1}$ are $\mathbb{N}$-independent (resp. $\mathbb{Z}$-independent). □■

5.3.3 Kovalevskaya exponents and Darboux polynomials

The same type of analysis can be carried out for Darboux polynomials. Recall that a Darboux polynomial P for a polynomial vector field $\delta_{\mathbf{f}}$ is a polynomial second integral, that is,

$$\delta_{\mathbf{f}} P = \lambda P, \tag{5.44}$$

for some non-constant polynomial λ. The analysis of the relationship between Kovalevskaya exponents and the degree of Darboux polynomials was first performed by Moulin-Ollagnier *et al.* (1995). Their results are obtained in a completely different setting and illustrated in many examples. It was later discovered that the same type of analysis had been carried out almost a century earlier by the Russian mathematician Mikhail Nikolaevich Lagutinskii (see Dobrovol'skii *et al.* (1998) for an interesting historical discussion).

Theorem 5.7 *If a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a Darboux polynomial J such that $\delta_{\mathbf{f}} P = \lambda P$ and a dominant balance $\hat{\mathbf{x}} = \boldsymbol{\alpha} t^{\mathbf{p}}$ with Kovalevskaya exponents $\boldsymbol{\rho} = (-1, \mathbf{r})$ ($\mathbf{r} \in \mathbb{C}^{n-1}$), then there exists a vector $\mathbf{m}$ of positive integers such that*

$$\mathbf{r}.\mathbf{m} = d + \lambda^{(0)}(\boldsymbol{\alpha}), \tag{5.45}$$

where $d = \deg(P^{(0)})$, $\lambda^{(0)}$ is the weight-homogeneous component of λ of highest weighted degree and $|\mathbf{m}| \leq \deg(P^{(0)})$.

Proof. If P is a Darboux polynomial of $\delta_{\mathbf{f}}$ and there is a dominant balance $\hat{\mathbf{x}} = \boldsymbol{\alpha} t^{\mathbf{p}}$, then

$$\delta_{\mathbf{f}^{(0)}} P^{(0)} = \lambda^{(0)} P^{(0)}, \tag{5.46}$$

where $\mathbf{f}^{(0)}, P^{(0)}$ and $\lambda^{(0)}$ are the weight-homogeneous components of $\mathbf{f}, P$ and λ of highest weighted degree. Therefore, as before, we can restrict the analysis of the Darboux polynomials to their weight-homogeneous components. We follow the steps of the two previous theorems and consider the companion system of $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ around the fixed point $\mathbf{X}_*$, that is, the system $\mathbf{Y}' = \mathbf{F}(\mathbf{Y}) = \text{diag}(-1, \mathbf{r}, 1)\mathbf{Y} + \mathbf{G}(\mathbf{Y})$ around $\mathbf{Y} = 0$ where $\mathbf{X} - \mathbf{X}_* = M\mathbf{Y}$ and $\mathbf{Y} = (Y_1, \hat{\mathbf{Y}}, Y_N = Z)$. The polynomials $P^{(0)}$ and $\lambda^{(0)}$ expressed in the new variables $\mathbf{Y}$ reads

$$P^{(0)} = Z^{-d} P^{(0)}(\mathbf{X}_* + M\mathbf{Y}) = Z^{-d} \left(\hat{\mathbf{Y}}^{\mathbf{m}} + \sum_{\mathbf{n}, |\mathbf{m}| \leq |\mathbf{n}| \leq \deg(P)} a_{\mathbf{n}} \hat{\mathbf{Y}}^{\mathbf{n}} \right). \tag{5.47}$$

In the same way, we have

$$\lambda^{(0)} = Z^{-1} \lambda^{(0)}(\mathbf{X}_* + M\mathbf{Y}) = Z^{-1} \left(\lambda^{(0)}(\boldsymbol{\alpha}) + \sum_{\mathbf{n}, |\mathbf{n}| > 0} b_{\mathbf{n}} \hat{\mathbf{Y}}^{n} \right). \tag{5.48}$$

The condition $\delta_{\mathbf{f}^{(0)}} P^{(0)} = \lambda^{(0)} P^{(0)}$ now reads $\delta_{\mathbf{F}} P^{(0)} = Z \lambda^{(0)} P^{(0)}$. If we gather the terms of lowest degree in $\hat{\mathbf{Y}}$ we find

$$\mathbf{r}.\mathbf{m} - d = \lambda^{(0)}(\boldsymbol{\alpha}), \tag{5.49}$$

and the result follows. □

5.3.4 Kovalevskaya exponents for Hamiltonian systems

The fundamental relationship between Kovalevskaya exponents and the degrees of first integrals cannot be used to prove the nonintegrability of Hamiltonian systems. Indeed, due to the parity of Kovalevskaya exponents (Proposition 5.7) there always exist resonance relations between Kovalevskaya exponents and further analysis is required. Although Yoshida's statement on the relationship between irrational Kovalevskaya exponents and nonintegrability was not correct in full generality, Yoshida managed to prove it in a particular case using Ziglin's theory of nonintegrability (Ziglin, 1983a; Ziglin, 1983b). A complete discussion of Ziglin's result is given in Chapter 6. Yoshida studied the case of n-degree-of-freedom Hamiltonian systems with a diagonal kinetic contribution and an homogeneous potential of the form

$$H = \frac{1}{2}(p_1^2 + \ldots + p_n^2) + V(q_1, \ldots, q_n), \tag{5.50}$$

where $V(\mathbf{x})$ is homogeneous of degree k but $k \neq 0, \pm 2$. The Kovalevskaya exponents always come by pairs $\rho_i + \rho_{i+n} = \frac{k+2}{k-2}$, so we can define the difference between two exponents of each pair as $\Delta\rho_i = \rho_{i+n} - \rho_i$.

Theorem 5.8 (Yoshida, 1989) *If the n numbers $\{\Delta\rho_1, \Delta\rho_2, \ldots, \Delta\rho_n\}$ are $\mathbb{Q}$-independent, then the Hamiltonian system has no additional first integral beside the Hamiltonian itself.*

As a corollary, for the case of planar Hamiltonian systems with homogeneous potential, we have the following result: if the Hamiltonian $H = \frac{1}{2}(p_1^2 + p_2^2) + V(x_1, x_2)$ possesses a second first integral, then the Kovalevskaya exponents are rational.

5.4 Complete integrability and resonances

The analysis in the two previous sections was centered around two particular solutions, the constant solution $\hat{\mathbf{x}} = \mathbf{x}_*$ provided by the fixed points and the similarity invariant solution $\hat{\mathbf{x}} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$. The analysis of the vector field around these two types of solutions lead to a relationship between the degrees of the first integrals and both their linear eigenvalues and Kovalevskaya exponents. We continue our investigation here by looking at the vector field around the local general solution about the fixed point.

Theorem 5.9 *Consider a vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Let $\mathbf{x}_*$ be a fixed point around which the Jacobian matrix is semi-simple with linear eigenvalues $\boldsymbol{\lambda}$. If $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is completely algebraically integrable then*
(1) the linear eigenvalues are rationally related (i.e., there exists λ such that $\lambda_i = q_i\lambda$, $q_i \in \mathbb{Q}$), and
(2) $\delta_{\mathbf{f}} = \mathbf{f}\partial_{\mathbf{x}}$ is (formally) linearizable around $\mathbf{x}_*$.

In particular, this theorem implies that all of the local series around the fixed point are pure series in exponentials of t (see Section 3.8.5).

Proof. (1) Since there exist $(n-1)$ first integrals, from Theorem 5.2, there are $(n-1)$ linearly independent integer vectors $\{\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(n-1)}\}$ such that $\boldsymbol{\lambda}.\mathbf{m}^{(i)} = 0$. Hence, there exists λ such that $\lambda_i = q_i\lambda$, $q_i \in \mathbb{Q}$, $i = 1, \dots, n$.
(2) Without loss of generality, assume that the system is written in the variables $\mathbf{x}$ such that $\mathbf{x}_* = 0$ and the linear part is diagonal. Moreover, assume it has $(n-1)$ first integrals, $I_1, \dots, I_{n-1}$. We can evaluate these first integrals on the general local solutions $\mathbf{x} = \mathbf{P}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$ reordered in powers of t. That is, we write,

$$\mathbf{x}(t) = \sum_{i=1}^{\infty} \boldsymbol{\Psi}^{(i)} t^i, \tag{5.51}$$

where $\boldsymbol{\Psi}^{(i)} = \boldsymbol{\Psi}^{(i)}(C_1 e^{\lambda_1 t}, \dots, C_n e^{\lambda_n t})$ is a formal power series in its arguments with constant coefficients. Assume, by contradiction, that $\boldsymbol{\Psi}^{(1)} \neq \mathbf{0}$. Then, to second order in t, we have

$$I_i(\mathbf{x}) = I_i(\boldsymbol{\Psi}^{(0)}) + t\partial_{\mathbf{x}} I_i(\boldsymbol{\Psi}^{(0)}).\boldsymbol{\Psi}^{(1)} + \mathrm{O}(t^2). \tag{5.52}$$

Since I_i is constant, the last relation implies

$$I_i(\boldsymbol{\Psi}^{(0)}) = C_i, \tag{5.53.a}$$

$$\partial_{\mathbf{x}} I_i(\boldsymbol{\Psi}^{(0)}).\boldsymbol{\Psi}^{(1)} = 0, \qquad i = 1, \dots, n-1. \tag{5.53.b}$$

The first integrals I_i are functionally independent, therefore there exist values of the n arbitrary constants appearing in $\boldsymbol{\Psi}^{(0)}$ such that the gradients of I_i evaluated on $\boldsymbol{\Psi}^{(0)}$, are linearly independent. Since $\boldsymbol{\Psi}^{(1)} \neq 0$, (5.53.b) implies that $\boldsymbol{\Psi}^{(1)}$ is tangent to the flow, that is, $\boldsymbol{\Psi}_1 = K(t)\mathbf{f}(\boldsymbol{\Psi}^{(0)})$. However, since $\mathbf{x}(t)$ is a solution we have $\dot{\boldsymbol{\Psi}}^{(0)} + \boldsymbol{\Psi}_1 = \mathbf{f}(\boldsymbol{\Psi}^{(0)})$. Using $\boldsymbol{\Psi}^{(1)} = K(t)\mathbf{f}(\boldsymbol{\Psi}^{(0)})$, we have $\dot{\boldsymbol{\Psi}}^{(0)} = (1-K)\mathbf{f}(\boldsymbol{\Psi}^{(0)})$. This, however is not possible and gives a contradiction since, to lowest order, $\Psi_i^{(0)} = C_i e^{\lambda_i t} + \dots$, which implies $K = 0$. We conclude that $\boldsymbol{\Psi}^{(1)} = 0$. The same argument applies to every order. Assume that $\boldsymbol{\Psi}^{(i)} = 0$ for all $0 < i < k$, then we find $\boldsymbol{\Psi}^{(k)} = K(t)\mathbf{f}(\boldsymbol{\Psi}^{(0)})$ and conclude that $\dot{\boldsymbol{\Psi}}^{(0)} = (1 - kK(t))\mathbf{f}(\boldsymbol{\Psi}^{(0)})$, which in turn implies $K(t) = 0$. Since the local solutions are pure series, we conclude from Lemma 3.5 that the fixed point is formally linearizable. □

5.5 Complete integrability and logarithmic branch points

In the previous section, we studied the variational equation around the local general solutions in a neighborhood of a fixed point. We now study the variational equations around the local general solutions that exist around movable singularities. We already found necessary conditions for a weight-homogeneous system to be integrable. The conditions are simply given in terms of the the Kovalevskaya exponents. More precisely, the maximum number of independent algebraic first integrals is given by the dimension of the vector space spanned by the

Kovalevskaya exponents over the integers (the positive integers for polynomial first integrals). We can go one step further in our analysis and show that algebraic integrability implies that all solutions can be expanded in Puiseux series (see Ishii (1990) for similar results for nth order differential equations).

Theorem 5.10 *Consider a vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Assume that it has a balance $\{\boldsymbol{\alpha}, \mathbf{p} \in \mathbb{Q}^n\}$ for which the Kovalevskaya matrix is semi-simple. If $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is completely integrable then the local general series are Puiseux series.*

The essence of this theorem is that the existence of $(n-1)$ algebraic integrals implies that the local series are free of movable logarithmic branch points.

Proof. From Corollary 5.4, we know that all Kovalevskaya exponents are rational. We can evaluate the first integrals on the local series written as

$$\mathbf{x}(\tau) = \sum_{i=1}^{\infty} \boldsymbol{\Psi}^{(i)} Z^i, \tag{5.54}$$

where $Z = \log(\tau)$. The functions $\boldsymbol{\Psi}^{(i)}$ are of the form $\boldsymbol{\Psi}^{(i)} = \tau^{\mathbf{p}} \mathbf{P}^{(i)}(\mathbf{C}\tau^{\boldsymbol{\rho}}, \tau^q)$ where $\mathbf{P}^{(i)}(.)$ are formal power series with constant coefficients. Assume, by contradiction, that $\boldsymbol{\Psi}^{(1)} \neq \mathbf{0}$. Then, to second order in Z we have

$$I_i(\mathbf{x}) = I_i(\boldsymbol{\Psi}^{(0)}) + Z\partial_{\mathbf{x}} I_i(\boldsymbol{\Psi}^{(0)}).\boldsymbol{\Psi}^{(1)} + \mathrm{O}(Z^2). \tag{5.55}$$

Since I_i is constant, it implies

$$I_i(\boldsymbol{\Psi}^{(0)}) = C_i, \tag{5.56.a}$$

$$\partial_{\mathbf{x}} I_i(\boldsymbol{\Psi}^{(0)}).\boldsymbol{\Psi}^{(1)} = 0, \qquad i = 1, \dots, n-1. \tag{5.56.b}$$

The same argument used in the proof of Theorem 5.9 can be used to show that $\boldsymbol{\Psi}^{(i)} = \mathbf{0}$ for all i. That is, the local solutions are all pure Puiseux series. □

Example 5.8 A simple example. Consider the third order equation $\dddot{x} = \dot{x}x$ (Ishii, 1990). Written as a system, it reads

$$\dot{x}_1 = x_2, \tag{5.57.a}$$

$$\dot{x}_2 = x_3, \tag{5.57.b}$$

$$\dot{x}_3 = x_1 x_2. \tag{5.57.c}$$

This equation admits the two polynomial first integrals

$$I_1 = x_3 - \frac{1}{2}x_1^2, \quad I_2 = x_3 x_1 - \frac{1}{2}x_2^2 - \frac{1}{3}x_1^3. \tag{5.58}$$

Hence, we conclude that the local series are all pure Puiseux expansions. Moreover, since the dominant balance is such that $\mathbf{p} = (-2, -3, -4)$ and $\boldsymbol{\alpha} = (36, -72, 216)$, the local series are pure Laurent series and the equation passes Painlevé test #3.

Example 5.9 The Lorenz system. The system reads

$$\dot{x} = \sigma(y - x), \tag{5.59.a}$$

$$\dot{y} = \rho x - y - xz, \tag{5.59.b}$$

$$\dot{z} = xy - \beta z, \tag{5.59.c}$$

where $x, y, z, \sigma, \beta, \rho \in \mathbb{R}$. The Lorenz system has two principal balances with $\boldsymbol{\alpha} = (\pm 2i, \mp 2i/\sigma, -2/\sigma)$ and $\mathbf{p} = (-1, -2, -2)$. The Kovalevskaya exponents are $\boldsymbol{\rho} = (-1, 2, 4)$.

Proposition 5.10 *If $\{\beta, \sigma, \rho\} \neq \{1, \frac{1}{2}, 0\}$, the Lorenz system is not algebraically integrable (with two first integrals).*

Proof. There is one set of parameter values for which the system has two first integrals, namely $\{\beta, \sigma, \rho\} = \{1, \frac{1}{2}, 0\}$ with first integrals

$$I_1 = (x^2 - 2\sigma z)e^t, \quad I_2 = (y^2 + z^2)e^{2t}, \tag{5.60}$$

and the solutions can be expressed in terms of Jacobi elliptic functions. In addition, there exist two sets of values for which the system satisfies the Painlevé test: $\{\beta, \sigma, \rho\} = \{1, 2, 1/9\}$ with one time-dependent integral

$$I_3 = (x^2 - 2\sigma z)e^{2\sigma t}, \tag{5.61}$$

and $\{\beta, \sigma, \rho\} = \{0, \frac{1}{3}, \rho\}$, with the time-dependent integral

$$I_4 = \left(-\rho x^2 + \frac{1}{3}y^2 + \frac{2}{3}xy + x^2 z - \frac{3}{4}x^4\right) e^{\frac{4}{3}t}. \tag{5.62}$$

In the last two cases, the system can be further reduced and integrated exactly. For all other values of the parameters, the system does not satisfy the Painlevé test. The result follows from Theorem 5.10. □■

5.6 Multivalued first integral and local solutions

In the previous section, we showed that the existence of sufficiently many single-valued first integrals implies that the solution only exhibit algebraic movable singularities. We now consider systems which admit at least one multi-valued first integral (Ishii, 1992).

Example 5.10 The planar Lotka-Volterra system

$$\dot{x}_1 = \lambda_1 x_1 + a x_1 x_2, \tag{5.63.a}$$
$$\dot{x}_2 = \lambda_2 x_2 + b x_1 x_2, \tag{5.63.b}$$

has a logarithmic first integral

$$I = b x_1 + a x_2 - \lambda_2 \log x_1 + \lambda_1 \log x_2. \tag{5.64}$$

Equivalently, we can take the exponential of I to obtain

$$J = \exp(I) = e^{b x_1 + a x_2} x_1^{-\lambda_2} x_2^{\lambda_1}. \tag{5.65}$$

Let $\hat{\mathbf{x}}$ be any solution of the system. The constant associated with the first integral I is $\kappa_0 = I(\hat{\mathbf{x}})$ but

$$\kappa_{nm} = \kappa_0 + 2\pi i(\lambda_1 m + \lambda_2 n), \qquad m, n \in \mathbb{Z}, \tag{5.66}$$

is also a constant for the same solution. Therefore, one solution corresponds to infinitely many arbitrary constants. Conversely, consider $J = \kappa$ and the set of points $\mathbf{x}_{klm} \in \mathbb{C}^2$ that satisfy

$$b x_{1klm} + a x_{2klm} - \lambda_2 \log x_{1klm} + \lambda_1 \log x_{2klm} + 2\pi i(ak + bl + m) = \log \kappa. \tag{5.67}$$

Then, $J = \kappa$ on any solution curve through the point $\mathbf{x}_{klm}$. Therefore, one constant κ is connected to infinitely many solution curves. Moreover, it is possible to choose a and b such that the solutions satisfying $J = \kappa$ densely fill a subspace of $\mathbb{C}^2$ (Kruskal & Clarkson, 1992). The local solutions around a movable singularities exhibit a logarithmic branch point except when $\lambda_1 = \lambda_2 = \lambda$ and $a = b$, in which case there is a linear first integral $I = (x_1 + x_2)\exp(-\lambda t)$. ■

This last example seems to indicate that the multi-valuedness of the first integral implies the multi-valuedness of the local solutions around the movable singularities.

Theorem 5.11 (Ishii, 1992) *Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}^n$ and assume it has both a first integral $I = I(\mathbf{x})$ and a dominant balance $\mathbf{x} = \boldsymbol{\alpha} t^{\mathbf{p}}$. Let q be the least common denominator of $\mathbf{p}$, and define $\hat{\mathbf{x}} = \boldsymbol{\beta}\tau^{q\mathbf{p}}$ where $\tau = t^{1/q}$. If the first integral I is such that $I(\hat{\mathbf{x}})$ is not single-valued, then the dominant balance does not define a pure Puiseux expansion with $(n-1)$ free arbitrary constants. If, moreover, two different branches $I^{(1)}, I^{(2)}$ of $I(\hat{\mathbf{x}})$ are such that $\Delta I = I^{(1)}(\hat{\mathbf{x}}) - I^{(2)}(\hat{\mathbf{x}}) \neq 0$, then there is no pure Puiseux series with $(n-1)$ free arbitrary constants.*

In particular, the condition that $I(\hat{\mathbf{x}})$ is not single-valued implies that a multi-valued first integral cannot be composed of other first integrals. That is, if I is a polynomial first integral, then $\log(I)$ is a multi-valued first integral. However, $\log(I(\boldsymbol{\alpha} t^{\mathbf{p}}))$ is not multi-valued.

Example 5.11 The system

$$\dot{x}_1 = x_1(x_2+1)(x_3+a), \tag{5.68.a}$$
$$\dot{x}_2 = x_2(x_3+a)(x_1+1), \tag{5.68.b}$$
$$\dot{x}_3 = x_3(x_1+1)(x_2+1), \tag{5.68.c}$$

has two first integrals,

$$I_1 = x_1 - x_2 + \log x_1 - \log x_2, \tag{5.69.a}$$
$$I_2 = x_2 - x_3 + \log x_2 - a \log x_3, \tag{5.69.b}$$

and a dominant balance with $\mathbf{p} = (-1/2, -1/2, -1/2)$ and $\boldsymbol{\alpha} = i\frac{\sqrt{2}}{2}(1,1,1)$. Then, $q = 2$ and $\hat{\mathbf{x}} = (\beta_1\tau^{-1}, \beta_2\tau^{-1})$. We evaluate the first integrals on $\hat{\mathbf{x}}$ to obtain

$$I_1(\hat{\mathbf{x}}) = (\beta_1 - \beta_2)\tau^{-1} + \log\beta_1 - \log\beta_2, \tag{5.70.a}$$
$$I_2(\hat{\mathbf{x}}) = (\beta_2 - \beta_3)\tau^{-1} + \log\beta_2 - a\log\beta_3 - (1-a)\log\tau. \tag{5.70.b}$$

Here, $I_1(\hat{\mathbf{x}})$ is single-valued but $I_2(\hat{\mathbf{x}})$ is multi-valued with a logarithmic branch point at $\tau = 0$. Moreover, $\Delta I_2 = 2\pi i(a-1)$ and we conclude that the system does not have a pure Puiseux series with 2 free arbitrary constants. ■

The multivaluedness of the first integral is observed in the complex space $\mathbb{C}^n$. If we restrict the dynamics of the system to real phase space, multivaluedness does not occur and the first integral can become single-valued. For instance, consider the logarithmic first integral

$$I = J_0 + \sum_{i=1}^{s} c_i \log J_i, \tag{5.71}$$

where J_i are algebraic functions. The level sets $S_k = \{\mathbf{x} \in \mathbb{C}^n | J_k(\mathbf{x}) = 0\}$ have real dimension $(n-1)$ and are invariant under the flow (J_k is a second integral of the system). The real phase space $\mathbb{R}^n$ can be decomposed into the union of disjoint sets where the first integral I is single-valued. In particular, irregular motion and chaos is not, in general, a consequence of the multi-valuedness of the solution since if enough multi-valued first integrals are known, the system behaves regularly in the decomposed real regions.

5.7 Partial integrability

5.7.1 A natural arbitrary parameter

The theory developed in the previous sections is only applicable for complete algebraic integrability. As a consequence, it cannot directly be applied to Hamiltonian systems, since for most of the Liouville integrable systems, only half of the first integrals are algebraic. The integrability conditions related to Kovalevskaya exponents are based on scale-invariant systems. However, most of the systems do not exhibit such a scaling property. For instance, as soon as dissipation or damping is included under the form of linear terms, a system will loose its scale-invariance. In turn, these terms may destroy the integrability of the weight-homogeneous system. Nevertheless, scale-invariant systems can be considered as the first order systems in a perturbation expansion based on the scale-invariance. Based on this idea, we can decompose the problem of finding necessary conditions for the existence of k first integrals ($k \leq n-1$) into two parts. The first part of the analysis consists of finding conditions for the existence of k first integrals of the highest components of all weight-homogeneous decompositions of the vector field. Consider again a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}^n$ where $\mathbf{f}$ is analytic and assume that there exists a weight-homogeneous decomposition related to the exponent $\mathbf{p}$. That is,

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}^{(0)}(\mathbf{x}) + \ldots + \mathbf{f}^{(m)}(\mathbf{x}), \tag{5.72.a}$$

$$\mathbf{f}^{(i)}(\alpha^{-\mathbf{p}}x) = \alpha^{-\mathbf{p}-1+i}\mathbf{f}^{(i)}(\mathbf{x}). \tag{5.72.b}$$

The leading weight-homogeneous system $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ is scale-invariant. According to this scaling symmetry, any first integral can be decomposed into a finite sum of weight-homogeneous components (Proposition 2.3)

$$I(\mathbf{x}) = I^{(0)}(\mathbf{x}) + \epsilon I^{(1)}(\mathbf{x}) + \ldots, \tag{5.73.a}$$

$$I^{(i)}(\alpha^{-\mathbf{p}}\mathbf{x}) = \alpha^{d-i} I^{(i)}(\mathbf{x}), \tag{5.73.b}$$

where $\epsilon = \alpha^{-1}$ and $I^{(0)}(\mathbf{x})$ is a first integral of $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$. The parameter ϵ is an arbitrary parameter that does not appear in the vector field. Therefore, it can assume any value and is not necessarily small. Assume that there exist k first integrals for $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$. The second part of the analysis consists in deriving necessary conditions for the existence of first integrals for the non-homogeneous system. The general strategy to prove the partial integrability of a non weight-homogeneous system is as follows:

1. find necessary conditions for the existence of k ($0 < k < n$) first integrals of the weight-homogeneous vector field $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$;
2. build the k first integrals, $I_i^{(0)}$ ($i = 1, \ldots, k$);
3. find necessary conditions for the existence of l ($l \leq k$) first integrals for the non-homogeneous system, and
4. build the l first integrals J_i ($i = 1, \ldots, l$), each J_i is of the form

$$J = J^{(0)}(\mathbf{x}) + \epsilon J^{(1)}(\mathbf{x}, t) + \ldots, \tag{5.74.a}$$

$$J^{(0)} = P(I_1^{(0)}, \ldots, I_l^{(0)}), \tag{5.74.b}$$

   where P is a weight-homogeneous function with respect to the weighted degree of the first integrals $d_1, \ldots, d_k$.

5.7.2 Necessary conditions for partial integrability

We assume, using the results from the previous sections, the analysis of the leading weight-homogeneous component has already been performed and the explicit forms of the first integrals are known. That is, the vector field $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$ admits k independent first integrals $I_i = I_i(\mathbf{x})$, ($i = 1, \ldots, k$) of weighted degree d_i with respect to the weight $-\mathbf{p}$. We study the persistence of these first integrals, or homogeneous combinations of first integrals, when lower weight-homogeneous components are added to the system. Apply the scaling symmetry $\mathbf{x} \to \epsilon^{\mathbf{p}}\mathbf{x}$ and $t \to t/\epsilon$ to the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ to obtain

$$\dot{\mathbf{x}} = \sum_{i=0}^{m} \mathbf{f}^{(i)}(\mathbf{x})\epsilon^i. \tag{5.75}$$

The most general form that the first integrals can assume is given by

$$J(\mathbf{x}, t) = (J^{(0)}(\mathbf{x}) + \epsilon J^{(1)}(\mathbf{x}) + \ldots + \epsilon^l J^{(l)}(\mathbf{x}))h(t), \tag{5.76}$$

where $h(t)$ is such that $h(t) \to 1$ as $t \to 0$. By construction, $J^{(0)}$ is a function of the first integrals I_i,

$$J^{(0)} = P(I_1, \dots, I_k), \tag{5.77}$$

where P is weight-homogeneous with respect to the scaling $(I_i \to \epsilon^{d_i} I_i)$.

To derive necessary conditions for the existence of the first integral J, we evaluate the first integral on a local solution around the singularities to first order on ϵ. We assume that the weight-homogeneous system admits a pure Puiseux series $\mathbf{x}^{(0)}$ as a solution such that for open sets of the arbitrary constants the gradients of the first integrals I_i evaluated on $\mathbf{x}^{(0)}$ are linearly independent. Since the first integrals are explicitly known and singularity analysis can be performed on the leading weight-homogenous system, these conditions can be checked explicitly.

Lemma 5.1 *Let $\mathbf{x}^{(0)}$ be a pure Puiseux series solution of $\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x})$, then there exists a $\Psi - \epsilon$ expansion solution of (5.75) of the form*

$$\mathbf{x} = \mathbf{x}^{(0)} + \sum_{i=1}^{\infty} \epsilon^i \mathbf{x}^{(i)}, \tag{5.78.a}$$

$$\mathbf{x}^{(i)} = \sum_{j=0}^{i-l+1} \mathbf{s}_{ij} \left[\log(t - t_*)\right]^j, \tag{5.78.b}$$

where $\mathbf{s}_{ij}$ are convergent Puiseux series with finite principal parts..

The parameter l in (5.78.b) is the first order in ϵ where a logarithmic correction is required, so $\mathbf{x}$ reads

$$\mathbf{x} = \mathbf{s}_{00} + \epsilon \mathbf{s}_{10} + \epsilon^2 \mathbf{s}_{20} + \ldots + \epsilon^l (\mathbf{s}_{k0} + \mathbf{s}^{(l1)} \log(t - t_*)) + O(\epsilon^{l+1}). \tag{5.79}$$

The existence of this series and the convergence of all the Puiseux series appearing in (5.79) is given in a more general context in Chapter 7 (Proposition 7.1). We now give necessary conditions for partial integrability.

Proposition 5.11 *Let $I_1, \dots, I_k$ be k independent rational first integrals for the scale-invariant system $\dot{\mathbf{x}} = f^{(0)}(\mathbf{x})$ and assume that it admits a pure Puiseux series solution $\mathbf{s}^{(00)}$ such that $\partial_{\mathbf{x}} I_1(\mathbf{s}^{(00)}), \dots, \partial_{\mathbf{x}} I_k(\mathbf{s}^{(00)})$ are linearly independent. Let*

$$c_i = \partial_{\mathbf{x}} I_i(\mathbf{s}^{(00)}).\mathbf{s}^{(l1)}, \qquad i = 1, \dots, k, \tag{5.80}$$

where $\mathbf{s}^{(l1)}$ is defined in (5.79). If there exist k independent rational first integrals for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then

$$c_1 = c_2 = \ldots = c_k = 0. \tag{5.81}$$

Proof. We evaluate the first integral on the solution (5.79) and expand it in ϵ to obtain

$$\begin{aligned} J(\mathbf{x}) &= J^{(0)}(\mathbf{s}^{(00)}) \\ &+ \epsilon(\partial_{\mathbf{x}} J^{(0)}(\mathbf{s}^{(00)}).\mathbf{s}^{(10)} + J^{(1)}(\mathbf{s}^{(00)})) \\ &+ \ldots \\ &+ \epsilon^l \left[\partial_{\mathbf{x}} J^{(0)}(\mathbf{s}^{(00)}).(\mathbf{s}^{(l0)} + \log(t - t_*)\mathbf{s}^{(l1)} + \ldots)\right] \\ &+ O(\epsilon^{l+1}). \end{aligned} \tag{5.82}$$

Since $J(\mathbf{x})$ is constant on all solutions and the parameter ϵ is an arbitrary scaling parameter, each order in ϵ is constant in time. In particular, there is no logarithmic dependence. Hence, we have

$$\partial_{\mathbf{x}} J^{(0)}.s^{(l1)} = 0. \tag{5.83}$$

Now, consider the k first integrals, $J_i = J_i(\mathbf{x})$ $(i = 1, \dots, l)$. For each first integral, we have

$$\partial_{\mathbf{x}} J_i^{(0)}(\mathbf{s}^{(00)}).\mathbf{s}^{(l1)} = 0. \tag{5.84}$$

The functional form of $J_i^{(0)}$ is not known. However, $J_i^{(0)}$ is a function of the first integrals I_i $(i = 1, \dots, k)$ and

$$\partial_{\mathbf{x}} J^{(0)}(\mathbf{s}^{(00)}) = \sum_{j=1}^{l} A_{ij} \partial_{\mathbf{x}} I_j^{(0)}(\mathbf{s}^{(00)}), \tag{5.85}$$

with $A \in \mathrm{GL}(k, \mathbb{C})$, owing to the independence of the first integrals on the solution $\mathbf{s}^{(00)}$. Therefore, the integrability conditions (5.84) read

$$\sum_{j=1}^{l} A_{ij} \partial_{\mathbf{x}} I_j^{(0)}(\mathbf{s}^{(00)})\mathbf{s}^{(k1)} = \sum_{j=1}^{l} A_{ij} c_j = 0. \tag{5.86}$$

Since A is invertible, we have $c_i = 0$ for all $i = 1, \dots, l$. □

Example 5.12 A three-degree-of-freedom Hamiltonian. As a first example, consider the Hamiltonian (Meletlidou & Ichtiaroglou, 1994)

$$H = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2\right) + \frac{1}{4}\left(x_1^4 + x_2^4 + x_3^4\right) + \epsilon\left(\mu_1 x_2 x_3 + \mu_2 x_3 x_1 + \mu_3 x_1 x_2\right), \tag{5.87}$$

where $\mu_i \neq 0,\ i = 1, 2, 3$. For $\epsilon = 0$, the system is integrable with three first integrals, the Hamiltonians of the decoupled systems:

$$I_i^{(0)} = 2y_i^2 + x_i^4, \qquad i = 1, 2, 3, \tag{5.88}$$

and the solutions can be expanded in Laurent series $\mathbf{s} = (x_1, x_2, x_3, y_1, y_2, y_3)$ with

$$\mathbf{s}^{(00)} = (t - t_*)^p \left(\boldsymbol{\alpha} + \sum_{i=1}^{\infty} \mathbf{a}_i (t - t_*)^i\right), \tag{5.89}$$

where $\mathbf{p} = (-1, -1, -1, -2, -2, -2)$, $\mathbf{a}_i \in \mathbb{C}^6$ and $\mathbf{a}_4$ is arbitrary. The Kovalevskaya exponents are $\mathcal{R} = \{-1, -1, -1, 4, 4, 4\}$. It can be verified that $\partial_{\mathbf{x}} I_1^{(0)}(\boldsymbol{\alpha}), \partial_{\mathbf{x}} I_2^{(0)}(\boldsymbol{\alpha})$, and $\partial_{\mathbf{x}} I_3^{(0)}(\boldsymbol{\alpha})$ are linearly independent. Hence, $\partial_{\mathbf{x}} I_j^{(0)}(\mathbf{s}^{(00)})$ are also linearly independent. The series expansion for the perturbed problem is

$$\mathbf{x} = \mathbf{s}^{(00)} + \epsilon \mathbf{s}^{(10)} + \epsilon^2 \left(\mathbf{s}^{(20)} + \mathbf{s}^{(21)} \log(t - t_*)\right) + O(\epsilon^3), \tag{5.90}$$

where $(t - t_*)^{q_{ij}} \mathbf{s}^{(ij)} \in \mathbb{C}^6((t - t_*))$ for some integer q_{ij}. The first logarithmic contribution enters to order $O(\epsilon^2)$. The integrability conditions read

$$c_i = \partial_{\mathbf{x}} I_i^{(0)}.\mathbf{s}^{(21)}, \qquad i = 1, 2, 3. \tag{5.91}$$

The conditions $c_1 = c_2 = c_3 = 0$ give

$$2\mu_2\mu_3 - \mu_1\mu_3 - \mu_1\mu_2 = 0, \tag{5.92.a}$$
$$-\mu_2\mu_3 + 2\mu_1\mu_3 - \mu_1\mu_2 = 0, \tag{5.92.b}$$
$$-\mu_2\mu_3 - \mu_1\mu_3 + 2\mu_1\mu_2 = 0. \tag{5.92.c}$$

As a consequence, if $\mu_i \neq \mu_j$, where $i \neq j$, there exists at most one extra analytic or rational first integral besides the Hamiltonian. Furthermore, Meletlidou and Ichtiaroglou (1994) proved that the Hamiltonian (5.87) for $\mu_1 = 0, \mu_2 = \mu_3 = 1$ does not possess a second first integral. ■

Example 5.13 Another three-degree-of-freedom Hamiltonian. The second example is (Hietarinta, 1987)

$$H = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2\right) + \left(x_1 x_3^2 + x_2 x_3^2\right) + \epsilon\left(\mu_1 x_1^2 + \mu_2 x_2^2 + \mu_3 x_3^2\right), \tag{5.93}$$

where $\mu_i \neq 0,\ i = 1, 2, 3$. We first consider the weight-homogeneous system (obtained by setting $\epsilon = 0$). Is the system Liouville integrable? By inspection, a second first integral is easily found to be

$$I = p_1 - p_2. \tag{5.94}$$

A third first integral is lacking to complete the integration.

Lemma 5.2 *The Hamiltonian $H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \left(x_1 x_3^2 + x_2 x_3^2\right)$ is not Liouville integrable.*

Proof. We compute the Kovalevskaya exponents for the scale-invariant solution $\mathbf{x} = -3t^{-3}(t/2, t/2, -t, -1, -1, -2)$ and obtain

$$\rho \in \mathcal{R} = \{-1, 2, 3, 6, \frac{5 \pm i\sqrt{23}}{2}\}. \tag{5.95}$$

The occurrence of irrational Kovalevskaya exponents are incompatible with Liouville integrability for potentials with diagonal kinetic parts (see Theorem 5.8). □

Now, we consider the full system (5.93), and build the Ψ-ϵ expansion up to order $O(\epsilon)$

$$\mathbf{x} = \mathbf{s}^{(00)} + \epsilon(\mathbf{s}^{(10)} + \mathbf{s}^{(11)} \log(t - t_*)) + O(\epsilon^2), \tag{5.96}$$

where $\mathbf{x} = (x_1, x_2, x_3, y_1, y_2 y_3)$ and $\mathbf{s}^{(00)},\ \mathbf{s}^{(10)}$ are Laurent series. The series $\mathbf{s}^{(11)}$ reads

$$\mathbf{s}^{(11)} = -\frac{3}{4}(\mu_1 - \mu_2)(1, -1, 0, 0, 0, 0) + O((t - t_*)^2). \tag{5.97}$$

The conditions $\partial_{\mathbf{x}} H.\mathbf{s}^{(11)} = 0$ and $\partial_{\mathbf{x}} I.\mathbf{s}^{(11)} = 0$ lead to $\mu_1 = \mu_2$. In this case, the second constant of motion is found to be

$$I = (p_1^2 + \mu_1 x_1^2) + (p_2^2 + \mu_1 x_2^2) - 2(p_1 p_2 + \mu_1 x_1 x_2). \tag{5.98}$$

Therefore we have the following result.

Proposition 5.12 *The Hamiltonian system (5.93) does not have a second constant of motion unless $\mu_1 = \mu_2$.*

■

5.8 Exercises

5.1 In order to prove Proposition 5.6, introduce a new equation $\dot{x}_{n+1} = -\chi x_{n+1}$ and show that if this new system has a time-independent first integral of the form $I = x_{n+1} P(x_1, \ldots, x_n)$, then $\boldsymbol{\lambda}.\mathbf{m} = \chi$ for all sets of linear eigenvalues.

5.2 Consider the system

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{j=1}^{n} M_{ij} x_j, \qquad i = 1, \ldots, n, \tag{5.99}$$

where $M \in \mathrm{M}_n(\mathbb{C})$. *(i)* Assume that one of the column vectors of M is $\mathbb{N}$-independent and use Corollary 5.2 to prove that there is no formal first integral. *(ii)* If one of the column vectors of M is $\mathbb{Z}$-independent show that there is no rational first integral.

5.3 Consider the two-degree-of-freedom Hamiltonian (Kummer *et al.*, 1991)

$$H = -x_3 x_1^2 + a(x_2^2 - x_4^2)x_1 + x_3^3. \tag{5.100}$$

Show that it defines an homogeneous system and that there exists a second first integral $I = x_2^2 + x_4^2$. Find all balances and compute the corresponding Kovalevskaya exponents. Show that there exist values of a such that the Kovalevskaya exponents are irrational despite the Liouville integrability of the system.

5.4 Let $\delta_{\mathbf{g}}$ be a symmetry field of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, (that is, $\delta_{\mathbf{g}}\delta_{\mathbf{f}} - \delta_{\mathbf{f}}\delta_{\mathbf{g}} = 0$). Assume, moreover, that if $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$ is a dominant balance of $\delta_{\mathbf{f}}$ then *(i)* $\delta_{\mathbf{g}}$ is weight-homogeneous, *(ii)* $\mathbf{g}(\boldsymbol{\alpha}) \neq 0$, and, *(iii)* $\deg(g_i/x_i, \mathbf{p}) = \deg(g_j/x_j, \mathbf{p})$ for all i, j. Show that $\deg(g_i/x_i, -\mathbf{p})$ is a Kovalevskaya exponent (Kozlov, 1996; Borisov & Tsygvintsev, 1997). Show that if $\delta_{\mathbf{g}} = -\mathbf{p}\mathbf{x}\partial_{\mathbf{x}}$ is the Euler vector field of Proposition 2.2, then there is always a Kovalevskaya exponent equal to -1.

5.5 Assume that there exist two first integrals I_1, I_2 of the same weighted degree d for a scale-invariant system. Moreover, assume that there exists a dominant balance such that $\partial_{\mathbf{x}} I_1(\boldsymbol{\alpha})$ and $\partial_{\mathbf{x}} I_2(\boldsymbol{\alpha})$ are linearly independent. Show that d is a Kovalevskaya exponent of multiplicity of at least two.

5.6 Consider the second Halphen system (Maciejewski, 1998)

$$\dot{x}_1 = a_1 x_1^2 + (\lambda + a_1)\left(x_2 x_3 - x_1(x_2 + x_3)\right), \tag{5.101.a}$$

$$\dot{x}_2 = a_2 x_2^2 + (\lambda + a_2)\left(x_3 x_1 - x_2(x_3 + x_1)\right), \tag{5.101.b}$$

$$\dot{x}_3 = a_3 x_3^2 + (\lambda + a_3)\left(x_1 x_2 - x_3(x_1 + x_2)\right). \tag{5.101.c}$$

Compute the Kovalevskaya exponents and find conditions on the parameters for the non-existence of polynomial first integrals. Show that the results of this chapter cannot be applied to prove the non-existence of rational first integrals sand that the system is actually completely integrable for $-2\lambda = a_1 + a_2 + a_3$ with two rational first integrals

$$I_1 = \frac{x_1 - x_2}{x_2 - x_3}, \quad I_2 = \frac{x_2 - x_3}{x_3 - x_1}. \tag{5.102}$$

5.7 Consider the generalized Halphen system (Maciejewski, 1998)

$$\dot{x}_k = \sum_{i=1}^{n} (-1)^i x_{k+i-1} x_{k+1}, \qquad k = 1, \ldots, n, \tag{5.103}$$

with $x_{n+i} = x_i$. Show that for n even, the system is completely integrable with linear first integrals and that for odd $n \geq 3$, the system does not admit a polynomial first integral.

5.8 What can you say about the nonintegrability of the system

$$\dot{x}_i = x_i x_{i+1}, \qquad i = 1, \ldots, n, \tag{5.104}$$

with $x_{n+1} = x_1$? (Hint: compute the Kovalevskaya exponents of the system by using the matrix form of Chapter 4).

Chapter 6:

Hamiltonian systems

"The fluxion of a constant quantity is nothing."
Saunderson (1756)

Hamiltonian systems are an important class of dynamical systems. Due to their rich structure, most of the corresponding results on integrability and nonintegrability can be refined and adapted. The problem of integrability and nonintegrability in Hamiltonian systems appears already in the simplest systems. Therefore, the following discussion will be kept at an elementary level and will emphasize only those Hamiltonians globally defined in $\mathbb{R}^n$ rather than on those defined arbitrary manifolds endowed with a symplectic structure.

6.1 Hamiltonian systems

Hamiltonian systems describe, among others systems, the dynamics of mechanical systems. The equations of motion for these dynamics are systems of differential equations that can be obtained from a single function, the *Hamiltonian*. For given mechanical systems, Hamiltonians can be systematically derived from first principles, usually from the Lagrange description by a Legendre transformation.[1] It is not the purpose of this book to explain how such a derivation is performed and we refer the reader to standard textbooks in classical mechanics (Whittaker, 1944; Goldstein, 1980; Arnold, 1988a). Here, we assume that such a function is known and we discuss general integrability properties. We first consider the case of an *n-degree-of-freedom Hamiltonian system*. Let $H = H(\mathbf{p}, \mathbf{q})$ be a C^1 function of its argument with $(\mathbf{p}, \mathbf{q}) \in \mathbb{R}^{2n}$, then *Hamilton's equations* are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad i = 1, \dots, n, \tag{6.1.a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \dots, n. \tag{6.1.b}$$

The variables $\mathbf{q}$ and $\mathbf{p}$ are referred to as, respectively, the *positions* and the *momenta*. If we introduce $\mathbf{x} = (\mathbf{p}, \mathbf{q})$, then Hamilton's equation can be written in the compact form

$$\dot{\mathbf{x}} = J\partial_{\mathbf{x}} H, \tag{6.2}$$

where J is the symplectic matrix defined by

$$J = \begin{bmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{bmatrix}. \tag{6.3}$$

and where I_n is the $n \times n$ identity matrix.

[1]Note that not all Hamiltonians in classical mechanics arise from a Legendre transformation of the Lagrangian. For example, the problem of n point vortices in planar steady flow has a Hamiltonian description given by the stream function (Arnold *et al.*, 1997).

Example 6.1 Consider, for instance, n point particles with equal masses m moving in an n-dimensional space $\mathbf{q}$ and interacting through the potential $V = V(\mathbf{q})$. The momentum vector is $\mathbf{p} = m\dot{\mathbf{q}}$. The Hamiltonian of the system is then given by its total energy (kinetic plus potential). That is,

$$H = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{q}), \tag{6.4}$$

and Hamilton's equations are

$$\dot{q}_i = \frac{p_i}{m}, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \qquad i = 1, \ldots, n. \tag{6.5}$$

Note that these equations can also be written

$$m\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}, \tag{6.6}$$

and we recover Newton's equations for the motion of n interacting particles. ■

The symplectic matrix, J, can be used to define an operation on smooth functions. Let $A(\mathbf{x}), B(\mathbf{x})$ be two C^1 functions in $\mathbb{R}^{2n}$, then the *canonical Poisson bracket*Poisson bracket—textit of A and B is defined as

$$\begin{aligned} \{A(\mathbf{x}), B(\mathbf{x})\} &= -(\partial_{\mathbf{x}} A).(J\partial_{\mathbf{x}} B), \\ &= -\sum_{i,j=1}^{2n} \partial_{x_i} A J_{ij} \partial_{x_j} B. \end{aligned} \tag{6.7}$$

In terms of the variables $(\mathbf{p}, \mathbf{q})$, it reads

$$\{A(\mathbf{p}, \mathbf{q}), B(\mathbf{p}, \mathbf{q})\} = \sum_{i=1}^{n} \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right). \tag{6.8}$$

and it satisfies the following properties:

1. bilinearity,
2. skew-symmetry: $\{A, B\} = -\{B, A\}$,
3. the Leibniz rule: $\{A, BC\} = \{A, B\}C + \{A, C\}B$,
4. the Jacobi identity: $\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0$,
5. Non-degeneracy: if $\mathbf{x}_0$ is such that $\partial_{\mathbf{x}} A(\mathbf{x}_0) \neq \mathbf{0}$, then $\exists B$ such that $\{A, B\}(\mathbf{x}_0) \neq 0$.

Let $\mathbf{x} = \mathbf{x}(t)$ be a solution of Hamilton's equations. Then, the evolution in time of any smooth function $A = A(\mathbf{x}(t))$ is given by its Poisson bracket:

$$\begin{aligned} \dot{A} &= (\partial_{\mathbf{x}} A).\dot{\mathbf{x}}, \\ &= (\partial_{\mathbf{x}} A).(J\partial_{\mathbf{x}} H), \\ &= \{H, A\}. \end{aligned} \tag{6.9}$$

In particular, the Poisson bracket can be used to write system (6.2) as

$$\dot{x}_i = \{H, x_i\}, \qquad i = 1, \ldots, 2n. \tag{6.10}$$

Or, in compact form, $\dot{\mathbf{x}} = \{H, \mathbf{x}\}$. Similarly, we define the *Hamiltonian vector field* by $\delta_H = \{H, \cdot\}$.

Now, we consider the Hamiltonian system $\dot{\mathbf{x}} = \{H, \mathbf{x}\}$ and we apply an invertible smooth change of variables $\mathbf{y} = \mathbf{g}(\mathbf{x})$. The new system reads

$$\begin{aligned}\dot{\mathbf{y}} &= K\dot{\mathbf{x}},\\ &= KJ\partial_{\mathbf{x}}H(\mathbf{x}),\\ &= KJK^T\partial_{\mathbf{y}}H(\mathbf{y}),\end{aligned}$$

where $K = \partial_{\mathbf{x}}\mathbf{g}$ is the Jacobian matrix of the transformation. The new system, $\dot{\mathbf{y}} = KJK^T\partial_{\mathbf{y}}H(\mathbf{y})$, will be Hamiltonian with Hamiltonian $H(\mathbf{y})$ if and only if

$$KJK^T = J. \tag{6.11}$$

A transformation $\mathbf{y} = \mathbf{g}(\mathbf{x})$ such that $KJK^T = J$ is referred to as a *canonical transformation*. A canonical transformation preserves the Poisson bracket, that is, $\{A, B\}_{\mathbf{x}} = \{A, B\}_{\mathbf{y}}$.

We can also consider more general forms (*i.e., non-canonical*) for the Poisson bracket where property (5) may not hold. Let A, B be two smooth functions of $\mathbf{x} \in \mathbb{R}^n$ (where n is not necessarily even) and consider a *structure matrix* $J = J(\mathbf{x})$ depending on the variables $\mathbf{x}$. The Poisson bracket is then defined as

$$\{A(\mathbf{x}), B(\mathbf{x})\} = -(\partial_{\mathbf{x}}A).(J(\mathbf{x})\partial_{\mathbf{x}}B), \tag{6.12}$$

where J depends on $\mathbf{x}$ and satisfies $J^T = -J$ for all $\mathbf{x}$. The equations of motion now read

$$\dot{\mathbf{x}} = J(\mathbf{x})\partial_{\mathbf{x}}H. \tag{6.13}$$

Note that, in general, the non-degeneracy condition (5) does not hold (if, for instance, n is odd). When condition (5) is not satisfied, there exist functions A such that $\{A, B\} = 0$ for all B. These functions are called *Casimirs* of the system. This implies that for a given Poisson bracket, the Casimirs are first integrals for all Hamiltonian functions, and can be used to reduce the dynamics of the degenerate Hamiltonian system to the dynamics of another Hamiltonian system on a lower dimensional manifold. In the particular case where J is linear in $\mathbf{x}$, the Poisson bracket is called a *Lie-Poisson bracket*.

Example 6.2 The Euler system (2.16) can be cast in the form of a Hamiltonian system with a non-canonical Poisson bracket. Let J be the 6×6 structure matrix

$$J = \begin{bmatrix} S(\mathbf{x}-\mathbf{k}) & S(\mathbf{y}) \\ S(\mathbf{y}) & 0 \end{bmatrix}, \tag{6.14}$$

where $\mathbf{k}$ is a constant vector and $S(\cdot)$ is the skew-symmetric matrix associated with the cross-product in $\mathbb{R}^3$:

$$S(\boldsymbol{\alpha}) = \begin{bmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{bmatrix}. \tag{6.15}$$

Now, we define the Hamiltonian function

$$H = \frac{1}{2}({x_1}^2 + {x_2}^2 + {x_3}^2) + \frac{1}{2}(a_1{y_1}^2 + a_2{y_2}^2 + a_3{y_3}^2). \tag{6.16}$$

The vector field $\delta_H = \{H, \cdot\}$ can be identified with the vector field (2.16) by defining $I = \mathrm{diag}(I_1, I_2, I_3)$ as the matrix of principal moments of inertia and $\mathbf{x} = I\boldsymbol{\omega}$, $\mathbf{y} = \boldsymbol{\gamma}$, $\mathbf{k} = \mathbf{X}$ and $(a_1, a_2, a_3) = (I_1^{-1}, I_2^{-1}, I_3^{-1})$. The structure matrix (6.14) is of rank 4. Therefore, there exist two Casimirs, given by

$$C_1 = \mathbf{y}^2, \quad C_2 = \mathbf{x}\mathbf{y}. \tag{6.17}$$

Written in terms of Euler variables, we recognize the first integrals as that in (2.18). The first Casimir is due to the fact that $\boldsymbol{\gamma}$ is a unit vector $1 = \boldsymbol{\gamma}.\boldsymbol{\gamma}$ and the second represents the conservation of the vertical component of the angular momentum $C_2 = (I\boldsymbol{\omega}).\boldsymbol{\gamma}$. ∎

6.1.1 First integrals

Since the time evolution of a function A under the Hamiltonian flow H is given by $\dot{A} = \{H, A\}$, it is straightforward to see that a function $I = I(\mathbf{x}, t)$ is a *first integral of a Hamiltonian system* with Hamiltonian H if and only if

$$\partial_t I + \{H, I\} = 0. \tag{6.18}$$

We can now use Jacobi's identity to obtain Poisson's theorem (the proof is left as an exercise).

Theorem 6.1 (Poisson) *If I_1 and I_2 are two first integrals of a Hamiltonian system then $\{I_1, I_2\}$ is also a (possibly trivial) first integral.*

Poisson's theorem can be used to create new and possibly independent first integrals. Given two first integrals I_1, and I_2, $I_3 = \{I_1, I_2\}$ is a first integral as is $I_4 = \{I_2, I_3\}$ and so on. However, in most circumstances, the first integral I_3 is either trivial or functionally dependent on the others. Two time-independent first integrals I_1, I_2 are *in involution* if $\{I_1, I_2\} = 0$.

Example 6.3 Consider a three-degree-of-freedom Hamiltonian with a canonical Poisson bracket $H = H(\mathbf{q}, \mathbf{p})$. Assume that the two first components of the angular momentum are conserved. That is, $\{H, L_1\} = 0$ and $\{H, L_2\} = 0$ with

$$L_1 = q_2 p_3 - q_3 p_2, \quad L_2 = q_3 p_1 - q_1 p_3. \tag{6.19}$$

Then, by Poisson's theorem, we conclude that $\{L_1, L_2\} = L_3 = q_1 p_2 - q_2 p_1$ is also conserved. However, the three components of the angular momentum are not in involution. Nevertheless, we can define new first integrals in involution by considering functions of the angular momentum. Take for instance $I_1 = H$, $I_2 = L_1^2 + 2L_2^2$ and $I_3 = L_1^2 + 2L_3^2$. Clearly $\{I_1, I_2\} = \{I_1, I_3\} = 0$ and we can compute $\{I_2, I_3\}$ from repeated applications of properties (1-3) of the Poisson bracket

$$\begin{aligned}
\{I_2, I_3\} &= \{L_1^2 + 2L_2^2, L_1^2 + 2L_3^2\}, \\
&= 2\{L_1^2, L_3^2\} + 2\{L_2^2, L_1^2\} + 4\{L_2^2, L_3^2\}, \\
&= 8L_1L_3\{L_1, L_3\} + 8L_2L_1\{L_2, L_1\} + 16L_2L_3\{L_2, L_3\}, \\
&= 8\left(L_1L_3(-L_2) + L_2L_1(-L_3) + 2L_2L_3(L_1)\right), \\
&= 0.
\end{aligned} \tag{6.20}$$

Therefore, the Hamiltonian has three first integrals in involution and according to Arnold-Liouville's theorem which will be presented in the next section, the Hamiltonian is completely integrable and can be integrated by quadrature and is such that the motion is quasi-periodic (or periodic in degenerate cases). ■

In Chapter 2, we saw that a first integral can be used to reduce the dynamics of an n-dimensional system to a system evolving on a manifold of dimension $(n-1)$. The structure of Hamiltonian systems is such that a first integral different from a Casimir in a $2n$-dimensional phase space can be used to reduce the dynamics to a $(2n-2)$-dimensional phase space. To do so, consider a n-degree-of-freedom Hamiltonian system H with a non-degenerate Poisson bracket, and let $F = F(\mathbf{x})$ be a first integral. Assume that there is a point $\mathbf{x}_0 \in \mathbb{R}^{2n}$ such that $\partial_{\mathbf{x}} F(\mathbf{x}_0) \neq 0$. Following a similar construction as the one used in Proposition 2.9, in a neighborhood U of $\mathbf{x}_0$, we can introduce a set of canonical coordinates $\mathbf{y} = (\mathbf{p}, \mathbf{q})$ such that $p_n = F(\mathbf{p}, \mathbf{q})$.[2] Since the transformation is canonical in the new coordinates, the system is again Hamiltonian (with Hamiltonian $\tilde{H}$) and we have $\dot{p}_n = -\partial_{q_n} \tilde{H}(\mathbf{p}, \mathbf{q})$. However, $\dot{p}_n = 0$ and we conclude that $\tilde{H}$ does not depend on q_n. Therefore,

[2] First integrals are sometimes refer to as *isolating integrals* if there exists a coordinate system with $p_n = F(\mathbf{p}, \mathbf{q})$ and $\dot{q}_n = f(q_n)$ since they allow us to isolate the evolution of q_n on $F = const$ (Lichtenberg & Lieberman, 1983, p. 33). Casimirs, for instance, are not isolating first integrals.

Hamilton's equations decouple and the first $(n-1)$ pairs of equations read

$$\dot{q}_i = \frac{\partial \tilde{H}}{\partial p_i}(q_1,\ldots,q_{n-1},p_1,\ldots,p_{n-1},C),\ \ i=1,\ldots,n-1, \tag{6.21.a}$$

$$\dot{p}_i = -\frac{\partial \tilde{H}}{\partial q_i}(q_1,\ldots,q_{n-1},p_1,\ldots,p_{n-1},C),\ \ i=1,\ldots,n-1, \tag{6.21.b}$$

where $C = F(\mathbf{x})$ is the level set associated with the first integral F and C plays the role of a parameter in the new Hamiltonian $\tilde{H}$. Therefore, $\tilde{H}(q_1,\ldots,q_{n-1},p_1,\ldots,p_{n-1},C)$ is an $(n-1)$-degree-of-freedom Hamiltonian whose dynamics is defined in a $(2n-2)$-dimensional phase space. However, in order to explicitly introduce the new set of canonical coordinates, we have to solve another Hamiltonian system whose vector field δ_F is given by the first integral.

If more first integrals are known, we can repeat the previous reduction to lower dimensional manifold. If there are k first integrals, $I_1,\ldots,I_k$, in involution (that is, $\{I_i, I_j\} = 0\ \forall\, i,j$), then we can reduce the dynamics to a manifold of dimension $2(n-k)$. If $k = n-1$, then the system is reduced to a one-degree-of-freedom (hence, integrable) Hamiltonian system. The condition that the first integrals must be in involution is crucial for the existence of the canonical coordinates that allow us to integrate the system. This construction will be shown in the next section to be the essence of Liouville's theorem for integrable Hamiltonian.

In the nineteenth century, the quest for additional first integrals of autonomous Hamiltonian systems was one of the central problems of classical mechanics. To find first integrals of analytic or polynomial Hamiltonians, we must introduce the proper *ansatz*. If the Hamiltonian is polynomial, it is reasonable to look for polynomial first integrals, $I_d = I_d(\mathbf{x})$, of a given degree d and following the basic idea of Section 2.4, we find conditions on the coefficients of I_d by imposing $\{H, I_d\} = 0$. This leads to a set of algebraic equations for the coefficients. However, it was realized that most of the known integrals for classical systems are only *polynomial* in the momenta. Typically, first integrals found by separation of variables, that is, by finding a canonical transformation such that the new Hamiltonian decouples in a sum of independent Hamiltonians, are quadratic in the momenta whereas those found by Noether's theorem, which relates first integrals to one-parameter group of symmetries (Arnold, 1988a, p. 88), are linear in the momenta (this happens whenever the corresponding Lagrangian function is quadratic in the velocities). Therefore, rather than looking for first integrals that are polynomial in $\mathbf{p}$ and $\mathbf{q}$, we look for those that are polynomial in momenta with coefficients depending on $\mathbf{q}$. That is,

$$I_d = \sum_{\mathbf{i},|\mathbf{i}|\le d} a_{\mathbf{i}}(\mathbf{q})\mathbf{p}^{\mathbf{i}}. \tag{6.22}$$

The condition $\{H, I_d\} = 0$ now leads to a set of linear first order PDEs for the functions $a_{\mathbf{i}}(\mathbf{q})$. These integrals are usually referred to as *polynomial integrals of degree d* and the corresponding method to build them is known as *Whittaker's method*. Whittaker (1944) gives a description of first and second degree polynomial first integrals (Ankiewicz & Pask, 1983), however it has been applied to those of higher degree (Holt, 1982; Hietarinta, 1987; Mishra & Parashar, 1990; Cleary, 1989; Cleary, 1990). In particular, this method can be used to find the most general potentials which admit polynomial integrals of a given degree. However, it suffers from the same shortcoming as the method to find first integrals of general dynamical systems. That is, there are no a priori bounds on the degree of first integrals.

Example 6.4 The Holt Hamiltonian. We apply Whittaker's method to the Hamiltonian (Holt, 1982)

$$H = \frac{1}{2}(p_1^2 + p_2^2) + (q_1^2 + \alpha)q_2^{-\frac{2}{3}} + \beta q_2^{\frac{4}{3}}. \tag{6.23}$$

That is, we look for a first integral, cubic in the momenta, of the form

$$I = \sum_{i_1=0}^{3}\sum_{i_2=0}^{3-i_1} a_{i_1,i_2}(q_1,q_2)p_1^{i_1}p_2^{i_2}. \tag{6.24}$$

After equating the terms with the same power, the condition $\{H, I\} = 0$ leads to a system of 10 first order linear partial differential equations for the coefficients a_{i_1,i_2}. This system has a solution for $\beta = 3/4$ given by

$$I = 2p_1^3 + 3p_1p_2^2 + 3p_1\left(2(q_1^2+\alpha)q_2^{-\frac{2}{3}} - 3q_2^{\frac{4}{3}}\right) + 18p_2q_1q_2^{\frac{1}{3}}. \tag{6.25}$$

■

6.2 Complete integrability

6.2.1 Liouville integrability

Given an n-degree-of-freedom system with a canonical Poisson bracket, Liouville's theorem shows that the existence of n first integrals (including the Hamiltonian) in involution allows us to reduce the system to quadratures. That is, it allows us to express the general solution in terms of integrals (Perelomov, 1990).

Theorem 6.2 (Liouville) *Let $H = H(\mathbf{x}, t)$ be a Hamiltonian function defined on $\mathbb{R}^{2n}$ with a canonical Poisson bracket and assume that there exist n first integrals $F_1(\mathbf{x}, t), \dots, F_n(\mathbf{x}, t)$ in involution*

$$\{F_i, F_j\} = 0, \qquad i, j = 1, \dots, n. \tag{6.26}$$

If the integrals are independent on the level set

$$\mathcal{C}_{\mathbf{a}} = \{(\mathbf{x}, t) \in \mathbb{R}^{2n} \times \mathbb{R} : F_i = \mathbf{a},\ i = 1, \dots, n\}, \tag{6.27}$$

then the solutions of Hamilton's equation $\dot{\mathbf{x}} = \{H, \mathbf{x}\}$ on $\mathcal{C}_{\mathbf{a}}$ can be obtained by quadratures.

Hamiltonian systems satisfying the conditions of Liouville's theorem are known as *Liouville integrable systems*. Note that since all of the first integrals are in involution, the Hamiltonian vector field $\delta_{F_j} = \{F_j, \cdot\}$ is another integrable system which can be solved by quadrature. The main idea behind Liouville's theorem is that the first integrals can be used as local coordinates. The involution condition implies that the n vector fields δ_{F_j} commute with each other and provide a choice of canonical coordinates. In these coordinates, the Hamiltonian is reduced to a sum of n decoupled Hamiltonians that can be integrated. This theorem has been generalized for the case where the first integrals form a solvable Lie algebra and their Poisson brackets are linear in the first integrals (Kozlov, 1996, p. 76). It has also been generalized to more differential equations with Lie symmetries (Olver, 1993).

6.2.2 Arnold-Liouville integrability

The content of Liouville's theorem is purely analytic. It provides a general result on the solvability of the system which, in practical case, may be hard to achieve. Arnold subsequently generalized Liouville's theorem by giving it a geometric interpretation (Arnold, 1988a).

Theorem 6.3 (Arnold-Liouville) *Let $H = H(\mathbf{x})$ be a Hamiltonian function defined on $\mathbb{R}^{2n}$ with a canonical Poisson bracket, and assume that*

1. *there exist n smooth first integrals $F_1(\mathbf{x}), \dots, F_n(\mathbf{x})$ in involution:*

$$\{F_i, F_j\} = 0, \quad i, j = 1, \dots, n; \tag{6.28}$$

2. *the integrals are independent on the level set*

$$\mathcal{C}_{\mathbf{a}} = \{(\mathbf{x}, t) \in \mathbb{R}^{2n} \times \mathbb{R} : F_i = \mathbf{a},\ i = 1, \dots, n\}; \tag{6.29}$$

3. *and the vector fields δ_{F_j} are complete on $\mathcal{C}_{\mathbf{a}}$ (that is, the solutions of the system $\dot{\mathbf{x}} = \{F_j, \mathbf{x}\}$ with initial conditions $\mathbf{x}_0 \in \mathcal{C}_{\mathbf{a}}$ are bounded for all time).*

Then, the following are true:

1. *each connected component of $\mathcal{C}_{\mathbf{a}}$ is diffeomorphic to the product of a k-dimensional torus $\mathbb{T}^k$ with an $(n-k)$-dimensional Euclidean space $\mathbb{R}^{n-k}$ for some k. If, moreover, $\mathcal{C}_{\mathbf{a}}$ is compact, then $k = n$ and $\mathcal{C}_{\mathbf{a}}$ is diffeomorphic to a torus $\mathbb{T}^n$;*

2. *on $\mathbb{T}^k \times \mathbb{R}^{n-k}$, there exist coordinates $\varphi_1, \dots, \varphi_k$, and $z_1, \dots, z_{n-k}$ such that Hamilton's equations on $\mathcal{C}_{\mathbf{a}}$ are*

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}, \quad \dot{\mathbf{z}} = \mathbf{c}, \tag{6.30}$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{a})$ and $\mathbf{c}$ are constants.

An Hamiltonian that satisfies the conditions of Arnold's theorem is called *Arnold-Liouville integrable* or *completely integrable*. The theorem also implies that the n Hamiltonian systems, given by $H = F_i$, are also completely integrable.

A two-degree-of-freedom Hamiltonian system is completely integrable when an extra first integral is found since the the involution condition, $\{H, F\} = 0$, is automatically satisfied.

If the level sets of the first integrals form a compact manifold and other technical conditions are satisfied, a generalization of Arnold's theorem can be proved even in the case where the first integrals are not in involution (Fomenko, 1988, p. 165).

If the Poisson bracket for an n-dimensional system is degenerate and there exist k Casimirs, then on the reduced manifold of dimension $(n-k)$ with fixed Casimirs, the reduced Hamiltonian is non-degenerate and Arnold's theorem can be applied. Therefore, $(n-k)/2$ integrals in involution are required for complete integrability.

Example 6.5 We saw in Chapter 2 that in order for the six-dimensional Euler equations to be integrable, only 4 first integrals need to be found. To show this, we used Jacobi's last multiplier theorem which relates the existence of an invariant density to the last first integral. In the framework of Hamiltonian systems, the invariant density is directly related to the conservation of phase space volume under the flow. Euler's equations are Hamiltonian with a degenerate Poisson bracket (with 2 Casimirs as shown in Example 6.2). That is, we have $n = 6$ and $k = 2$. Therefore, two first integrals are needed for complete integrability, namely, the Hamiltonian itself and the "mathematical mermaid" of Section 2.1.1. ■

The variables $\boldsymbol{\varphi}, \mathbf{z}$ are not, in general, canonical variables. However, if $k = n$, there exists another set of canonical variables $(\mathbf{I}, \boldsymbol{\varphi})$ where $I_i = I_i(F_1, \dots, F_N)$. These variables known as *action-angle* variables can be computed by solving the so-called *Hamilton-Jacobi equation* (Arnold, 1988a, p. 255). In these variables, the new Hamiltonian is $H = H(I_1, \dots, I_n)$ and the equations of motion are

$$\dot{\mathbf{I}} = 0 \tag{6.31.a}$$

$$\dot{\boldsymbol{\varphi}} = \partial_{\mathbf{I}} H = \boldsymbol{\omega}(\mathbf{I}) \tag{6.31.b}$$

Therefore, the motion on the torus $\mathbf{I} = \mathbf{a}$ is, in general, quasi-periodic unless $(\omega_1, \dots, \omega_n)$ are rationally related, in which case, the torus is a *resonant torus*. If $\det(\partial_{\mathbf{I}} \boldsymbol{\omega}) \neq 0$ in the neighborhood of a torus, the Hamiltonian H is *non-degenerate* and almost all of the invariant tori are non-resonant.

If the number of first integrals is greater than the number of degrees of freedom, then not all the first integrals are in involution and the manifold $\mathcal{C}_{\mathbf{a}}$ is diffeomorphic to a torus $\mathbb{T}^k$ of dimension $k < n$ (Nekhoroshev, 1972). More precisely, when the conditions of Arnold's theorem are satisfied for a $2n$-dimensional non-degenerate Hamiltonian with $n + k$ independent first integrals, the manifold $\mathcal{C}_{\mathbf{a}}$ is diffeomorphic to a torus $\mathbb{T}^{n-k}$. These systems are sometimes refer to as *superintegrable* (Ranada, 1997). An important result related to superintegrable Hamiltonians is Bertrand's theorem which states that the only potentials in three dimensions with a diagonal kinetic part ($H = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{q})$) that have plane and periodic bounded orbits are the Kepler potential $V = -\frac{C}{|\mathbf{q}|}$ and the harmonic oscillator $V = C|\mathbf{q}|^2$. Both Hamiltonians have five independent first integrals in involution which implies that the solutions lie on a torus $\mathbb{T}^1$ and hence are periodic (This is the content of the cover picture on Goldstein's classical mechanics textbook (1980)).

6.3 Algebraic Integrability

In Chapter 5, we related the existence of first integrals to the existence of local Puiseux series. It was shown that if there are $(n-1)$ algebraic first integrals for an n-dimensional system, the solutions do not exhibit logarithmic branch points. However, for a Hamiltonian system, Liouville integrability cannot be related to the absence of logarithmic branch points since only $n/2$ first integrals are known. Yet, in most of the Hamiltonian examples presented in Chapter 2, 3 and 5, the Painlevé property seemed intimately connected to Liouville integrability (see for instance, the conditions for the existence of meromorphic solutions of the Euler equations in Chapter 3). Therefore, to relate the local analysis of solutions in complex time with the existence of first integrals a stronger definition of integrability has to be introduced for Hamiltonian systems. Basically, Liouville integrability relates the existence of first integrals to the property that the solutions lie on *real* invariant tori where the dynamics is linear. If these tori are the real part of *complex* tori and the complexified motion is linear then it is known that the solutions might be expressed in terms of Abelian functions (see below) which are meromorphic functions of time. This is precisely the motivation behind the work of Adler and van Moerbeke (1982a; 1982b; 1989a; 1989b) who introduced the notion of *algebraic complete integrability* that we will discuss now.

In order to introduce the notion of Abelian functions, we follow Dubrovin (1982) and Kozlov (1996) and consider a function $F = F(\mathbf{z}),\ \mathbf{z} \in \mathbb{C}^n$. This function is *meromorphic* if it can be written as the ratio of two convergent power series functions in the whole space, that is, $F = f(\mathbf{z})/g(\mathbf{z})$.

Definition 6.1 An *Abelian function* $F(\mathbf{z})$ is a meromorphic function in $\mathbb{C}^n$ with $2n$ $\mathbb{R}$-independent periods $(\boldsymbol{\omega}^{(1)}, \dots, \boldsymbol{\omega}^{(2n)})$. That is,

$$F(\mathbf{z} + \boldsymbol{\omega}^{(j)}) = F(\mathbf{z}),\ \forall\ \mathbf{z} \in \mathbb{C}^n, \quad j = 1, \dots, 2n. \tag{6.32}$$

The simplest such functions are the elliptic functions, that is, the set of doubly periodic functions in the complex plane. Now, since an Abelian function is periodic with $2n$ independent periods, it will take the same values on points that differ by an integer linear combination of the periods. These points can therefore be identified and such an identification is performed by introducing a fundamental lattice

$$\Gamma = \left\{ k_1\boldsymbol{\omega}^{(1)} + \dots + k_{2n}\boldsymbol{\omega}^{(2n)}, (k_1, \dots, k_{2n}) \in \mathbb{Z}^{2n} \right\}. \tag{6.33}$$

We proceed by defining two points $\mathbf{z}^{(1)}$ and $\mathbf{z}^{(2)}$ as *equivalent* if they differ by an element of the lattice $\mathbf{z}^{(1)} = \mathbf{z}^{(2)} + \boldsymbol{\gamma}$ for some $\boldsymbol{\gamma} \in \Gamma$. This equivalence is expressed by introducing the *quotient space* $\mathbb{C}^n/\Gamma$. The existence of $2n$ periods implies that this space is equivalent to a torus $\mathbb{T}^{2n}$, called an *Abelian torus*. The *Abelian functions* are the set of meromorphic functions on $\mathbb{T}^{2n}$. In order to define complete algebraically integrable systems (Adler & van Moerbeke, 1989a), we consider real Hamiltonian systems with non-canonical Poisson brackets of the form

$$\dot{\mathbf{x}} = \{H, \mathbf{x}\} = J(\mathbf{x})\partial_{\mathbf{x}} H, \quad \mathbf{x} \in \mathbb{R}^n, \tag{6.34}$$

where the structure matrix $J(\mathbf{x})$ and $H(\mathbf{x})$ are real polynomials in $\mathbf{x}$. Let $F_1, \dots, F_k$ be the polynomial Casimirs of J ($J\partial_{\mathbf{x}} F_i = 0,\ i = 1, \dots, k$) and assume that on the level sets of the Casimir functions, the Hamiltonian system is non-degenerate.

Definition 6.2 The Hamiltonian system (6.34) is *algebraically completely integrable (a.c.i.)* when:

1. in addition to the Casimirs $F_1, \dots, F_k$ the system has $m = (n-k)/2$ additional independent polynomial first integrals $F_{k+1}, \dots, F_{k+m}$ in involution such that for generic $\mathbf{a} = (a_1, \dots, a_{k+m}) \in \mathbb{R}^{k+m}$ the invariant manifolds

$$\mathcal{C}_{\mathbf{a}}^{\mathbb{R}} = \{\mathbf{x} \in \mathbb{R}^{2n} : F_i(\mathbf{x}) = a_i,\ i = 1, \dots, k+m\}, \tag{6.35}$$

are compact and connected and therefore, according to Arnold's theorem, diffeomorphic to real tori;

2. for a generic constant $\mathbf{a}$ there exists an Abelian torus $\mathbb{T}^{2m}$ with complex coordinates $\tau_1,\dots,\tau_m$ and n Abelian functions $\mathbf{z}=\mathbf{z}(\tau_1,\dots,\tau_m)$ parametrizing the non-compact invariant manifolds

$$\mathcal{C}_{\mathbf{a}}^{\mathbb{C}} = \{\mathbf{z} \in \mathbb{C}^{2n} : F_i(\mathbf{z}) = a_i,\ i = 1, \dots, k+m\}; \tag{6.36}$$

3. on $\mathcal{C}_{\mathbf{a}}^{\mathbb{C}}$, the complex Hamiltonian flow (that is, the system $\dot{\mathbf{z}} = \{H, \mathbf{z}\}$ with complex time t) is linear and can be written

$$\dot{\tau}_i = \mu_i, \qquad i = 1, \ldots, m, \tag{6.37}$$

where μ_i is constant for all i.

Therefore, the definition of a.c.i. systems imposes that for almost all values, the level sets of the first integrals over the real numbers are real algebraic tori and that each real torus is part of a complex Abelian torus on which the complexified phase flow of the system is linear. The fundamental theorem of Adler and van Moerbeke (1982a) relates a.c.i. systems to the meromorphicity of the local solutions.

Theorem 6.4 *If a system $\dot{\mathbf{x}} = \{H, \mathbf{x}\}$ is completely algebraically integrable with k Abelian functions, then the system admits local Laurent series solutions with ascending powers and $(k-1)$ free parameters (that is, $k-1$ positive integer Kovalevskaya exponents) of the form*

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{i=1}^{\infty} \mathbf{c}_i \tau^i \right), \tag{6.38}$$

where $\tau = (t - t_*)$, $-\mathbf{p} \in \mathbb{N}^n$ (that is, each component blows up), and $\mathbf{c}_i$ is constant for all i.

This theory has been successfully applied to the integrable cases of the Euler equations, some geodesic motions on $SO(4)$ and some generalizations of the Toda lattices (Adler & van Moerbeke, 1989a). The analysis of the local solutions provides an efficient way to derive necessary conditions for complete algebraic integrability and can be used for the construction of global action-angle variables for a.c.i. systems (Novikov & Veselov, 1985; Vanhaecke, 1992; Abenda, 1998).

6.4 Ziglin's theory of nonintegrability

In the previous sections, we established that in order for an n-degree-of-freedom Hamiltonian to be integrable, only n first integrals are needed. This general property prohibits us from using the general results on nonintegrability discussed in Chapter 5 which relate complete algebraic integrability (in this case $2n-1$ first integrals) with the existence of Puiseux series solutions. Moreover, due to the Hamiltonian structure, linear eigenvalues and Kovalevskaya exponents are always paired and are thus resonant. Therefore, the general results relating the degrees of first integrals to linear eigenvalues and Kovalevskaya exponents do not provide any relevant information (at best they provide an heuristic argument, namely, that the system may not be integrable when some of the Kovalevskaya exponents are irrational). Clearly, a new approach is needed. In Chapter 5, nonintegrability results were based on the analysis of the variational equation around local solutions (fixed points, movable singularities, general local solutions around the fixed point and general local solutions around a movable singularity). In order to obtain a criterion of nonintegrability for Hamiltonian systems, we can still use the variational equation but we need to consider a global solution. The advantage of this approach is that complete analytic integrability imposes strong conditions on the solutions of the variational equation around a globally defined function. A criterion, whenever applicable, can provide almost optimal results on nonintegrability (that is, it identifies all the cases when the system is nonintegrable). The drawback of this approach is in its global character. In general, it will not allow us to design a general algorithm to prove nonintegrability.

Ziglin's theory of nonintegrability considers the variational equations around a straight line solution in phase space. Let $H = H(\mathbf{x}),\ \mathbf{x} \in \mathbb{C}^{2n}$ be a Hamiltonian with a canonical Poisson bracket. We look for a straight line solution of the form

$$\sum_{i=1}^{2n} a_i x_i(t) = 0, \quad a_i \in \mathbb{C}, \tag{6.39}$$

and assume that along this line in a possibly complexified phase space, we can solve the system exactly to obtain the solution $\hat{\mathbf{x}} = \hat{\mathbf{x}}(t)$. The analysis of the variational equation around this particular solution leads to conditions for the existence of an additional analytic first integral. Rather than studying the problem in full generality, we first study in detail a simple example, then discuss the theory for two-degree-of-freedom Hamiltonians and its different extensions and applications.

Example 6.6 Yoshida's example (1986). Consider the homogeneous quartic Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}(q_1^4 + q_2^4) + \frac{a}{2}q_1^2 q_2^2. \tag{6.40}$$

The problem is to find all values of the parameter a for which the system does not admit a second analytic first integral. This system can be shown, by direct computation, to be integrable for

$$a = 0, \qquad I = \frac{1}{2}(p_1^2 - p_2^2) + \frac{1}{4}(q_1^4 - q_2^4), \tag{6.41.a}$$

$$a = 1, \qquad I = p_1 q_2 - p_2 q_1, \tag{6.41.b}$$

$$a = 3, \qquad I = p_1 p_2 + q_1 q_2 (q_1^2 + q_2^2). \tag{6.41.c}$$

The existence of these three values of a for which the system is integrable does not preclude the possible existence of other (isolated or not) values of a for which the system might be integrable with an analytic first integral.

Singularity analysis leads to 2 different sets of Kovalevskaya exponents, both corresponding to the dominant exponents $\mathbf{p} = (-1, -1, -2, -2)$. The first corresponds to a balance of order 2 (when either $q_1 = p_1 = 0$ or $q_2 = p_2 = 0$), and the second to a balance of order 4. The two sets of Kovalevskaya exponents are

$$\mathcal{R}_1 = \{-1, 4, \frac{3}{2} \pm \frac{1}{2}\sqrt{1 + 8a}\}, \tag{6.42.a}$$

$$\mathcal{R}_2 = \{-1, 4, \frac{3a + 3 \pm \sqrt{-7a^2 + 18a + 25}}{2(a+1)}\}. \tag{6.42.b}$$

The Kovalevskaya exponents are integer valued only if $a = 0, 1, 3$, that is, in the integrable cases. For these cases, we verify that the compatibility conditions are satisfied and that the local solutions are all Laurent series. However, at the level of integrability, the only conclusion that we can draw using the theorems of Chapter 5 is that when $a \notin \{0, 1, 3\}$, the system is not superintegrable (with 3 analytic first integrals).

In order to apply Ziglin's method, we look for straight line solutions of the Hamiltonian vector field

$$\dot{q}_1 = p_1, \tag{6.43.a}$$

$$\dot{q}_2 = p_2, \tag{6.43.b}$$

$$\dot{p}_1 = -q_1(q_1^2 + a q_2^2), \tag{6.43.c}$$

$$\dot{p}_2 = -q_2(q_2^2 + a q_1^2). \tag{6.43.d}$$

By inspection, we find that $q_1 = 0$ and $q_1 = q_2$ are two such solutions. We first study $q_1 = 0$. In this case, the equations decouple to

$$\ddot{q}_2 = -q_2^3, \tag{6.44}$$

which is solved by elliptic functions (Lang, 1987). Since we are trying to prove the non-existence of a second first integral for all values of H, we can choose a value of H for which the solution is particularly simple. For $H = 1/4$, we have

$$\hat{q}_2 = \mathrm{cn}(t, k), \tag{6.45}$$

where $k = \frac{1}{\sqrt{2}}$. The Jacobi elliptic cosine is a doubly periodic function with fundamental periods $T_1 = 4\mathrm{K}(k)$ and $T_2 = 4i\mathrm{K}(k')$ where K(k) is the complete elliptic integral of the first kind and $k' = \sqrt{1 - k^2}$. Now, consider the variational equation around the solution $\hat{\mathbf{x}} = (0, 0, \hat{q}_2, \frac{d\hat{q}_2}{dt})$. That is,

$$\dot{u}_1 = v_1, \tag{6.46.a}$$
$$\dot{u}_2 = v_2, \tag{6.46.b}$$
$$\dot{v}_1 = -a\hat{q}_2^2 u_1, \tag{6.46.c}$$
$$\dot{v}_2 = -3\hat{q}_2^2 u_2. \tag{6.46.d}$$

This system decouples into two second order equations. The second equation $\ddot{u}_2 + 3\hat{q}_2^3 u_2 = 0$, describes the variation tangent to the orbits defined by $\hat{q}_2$ and hence provides no information on the integrability of the original system. The first equation

$$\ddot{u}_1 + a\hat{q}_2^2 u_1 = 0, \tag{6.47}$$

is called the *normal variational equation* and contains valuable information. We now assume, by contradiction, that the original system has a second first integral I independent of H and show that for some values of a, the properties of the solution of the normal variational equation are inconsistent with the existence of I. If there exists a first integral I for the Hamiltonian system, then by Poincaré's Lemma 5.3, the quantity

$$J = \mathbf{u}.\partial_{\mathbf{x}} I(\hat{\mathbf{x}}), \tag{6.48}$$

where $\mathbf{u} = (u_1, u_2, v_1, v_2)$, is a first integral of the variational equation. However, if $\partial_{\mathbf{x}} I(\hat{\mathbf{x}}) = \mathbf{0}$, this integral is trivial and useless for the analysis. Nevertheless, the first major result of Ziglin (see Lemma 6.1) is that if I is analytic, then there is always a integer $k \geq 1$ such that $J_k = (\mathbf{u}.\partial_{\mathbf{x}})^k I(\hat{\mathbf{x}})$ does not vanish identically (with $J_j = 0, \;\; 0 < j < k$). Moreover, the first integral on the normal variational equation, obtained by setting $u_2 = v_2 = 0$, is also a non-trivial first integral. We conclude that the normal variational equation has an homogeneous first integral in u_1, v_1 given by

$$\begin{aligned} J_k &= (u_1\partial_{q_1} + v_1\partial_{p_1})^k I(\hat{\mathbf{x}}), \\ &= \sum_{i=0}^{k} c_i(\hat{\mathbf{x}}(t)) u_1^i v_1^{k-i}. \end{aligned} \tag{6.49}$$

Consider, the normal variational system

$$\dot{u}_1 = v_1, \tag{6.50.a}$$
$$\dot{v}_1 = -a\hat{q}_2^2 u_1, \tag{6.50.b}$$

and let $\mathbf{\Phi}(t)$ be a fundamental solution matrix. Fix an initial time $t = t_0$, and let the fundamental solution evolve for any period T of the function $\hat{q}_2$ to obtain $\mathbf{\Phi}(t_0 + T)$. This new matrix is also a fundamental solution and is related to $\mathbf{\Phi}(t_0)$ by

$$\mathbf{\Phi}(t_0 + T) = M(T)\mathbf{\Phi}(t_0), \tag{6.51}$$

where $M(T)$ is a *monodromy matrix* which is constant and such that $\det(M(T)) = 1$ (see the discussion of Section 3.5.3). Since the solution $\hat{q}_2$ has two independent periods, T_1 and T_2, two different matrices $M^{(1)} = M(T_1)$ and $M^{(2)} = M(T_2)$ can be associated with (6.51). Now, assume that one of these 2×2 matrices, say $M^{(1)}$, is *non-resonant*, that is, its eigenvalues ρ and $1/\rho$ are not roots of unity ($\rho^k \neq 1, \; \forall \; k \in \mathbb{Z}$). In particular, a non-resonant 2×2 matrix has different eigenvalues and can be diagonalized. Let C_0 be the value of the first integral J_k evaluated on an initial point t_0 and written in the variables (ξ, η) where $M^{(1)}$ is diagonal; that is,

$$C_0 = \sum_{i=0}^{k} d_i(\hat{\mathbf{x}}(t_0)) \xi(t_0)^i \eta(t_0)^{k-i}. \tag{6.52}$$

Consider the value, C_1, of the same first integral at time $t = t_0 + T_1$ and use the fact that $\hat{\mathbf{x}}$ is periodic, $\xi(t_0 + T_1) = \rho\xi(t_0)$ and $\eta(t_0 + T_1) = \rho^{-1}\eta(t_0)$ to obtain

$$C_1 = \sum_{i=0}^{k} d_i(\hat{\mathbf{x}}(t_0))\rho^{i-j}\xi(t_0)^i\eta(t_0)^{k-i}. \tag{6.53}$$

Since J_k is a first integral, we have $C_0 = C_1$ and conclude that $i = j$. That is, J_k has the form

$$J_k = d(\hat{\mathbf{x}}(t))(\xi\eta)^l. \tag{6.54}$$

Now, in the basis where $M^{(1)}$ is diagonal, the matrix $M^{(2)}$ has the general form

$$M^{(2)} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \tag{6.55}$$

with $\det(M^{(2)}) = 1$. Again, we fix a point t_0 and consider the first integral at $t = t_0$ and $t = t_0 + T_2$. Using (6.54) and the periodicity of the coefficients, we find

$$J_k = d(\hat{\mathbf{x}}(t_0))(\xi\eta)^l = d(\hat{\mathbf{x}}(t_0))\left[(\alpha\xi + \beta\eta)(\alpha\xi + \beta\eta)\right]^l. \tag{6.56}$$

That is, $\alpha\gamma - \beta\delta = 0$ and $\alpha\delta - \beta\gamma = 1$, which implies that either

1. $\beta = \gamma = 0$ (*i.e.,* $M^{(1)}$ and $M^{(2)}$ commute), or
2. $\alpha = \delta = 0$, (*i.e.,* $\mathrm{tr}(M^{(2)}) = 0$).

Therefore, we conclude that if there exists a second first integral and one of the monodromy matrices is non-resonant then either both matrices commute or the other matrix is traceless. In essence, this is the main statement of Ziglin's theorem (Ziglin, 1983a; Ziglin, 1983b).

The problem now is to compute, explicitly if possible, the monodromy matrices $M^{(1)}, M^{(2)}$ and check the conditions for one of the matrix to be non-resonant and for the other to commute with the first one or to be traceless. To do so, we consider the normal variational equation $\ddot{u}_1 + a\hat{q}_2^2 u_1 = 0$ and introduce the change of independent variable

$$\xi = u_1(z), \quad z = \hat{q}_2^3, \tag{6.57}$$

to obtain *Gauss hypergeometric equation*

$$z(1-z)\xi'' + \frac{3}{4}(1+z)\xi' + \frac{a}{8}\xi = 0. \tag{6.58}$$

It can be shown that the monodromy matrices $M^{(1,2)}$ can be expressed in terms of the product of the monodromy matrices, $M(\gamma_0)$ and $M(\gamma_1)$, of the hypergeometric equation obtained by taking a closed path, around $z = 0$ and $z = 1$ respectively. More precisely, $M^{(1)} = M(\gamma_0)M(\gamma_1)M(\gamma_0)$ and $M^{(2)} = M(\gamma_0)^2 M(\gamma_1)$. The form of these two monodromy matrices is explicitly known (Yoshida, 1986):

$$M(\gamma_0) = \begin{bmatrix} e^{\frac{\pi i}{2}} & 0 \\ e^{-\pi i\alpha_+} - e^{\frac{\pi i}{2}} & 1 \end{bmatrix}, \tag{6.59.a}$$

$$M(\gamma_1) = \begin{bmatrix} 1 & 1 - e^{-\pi i(\frac{1}{2}+\alpha_-)} \\ 0 & -1 \end{bmatrix}, \tag{6.59.b}$$

where $\alpha_\pm = \frac{1}{4} \pm \frac{\sqrt{1+8a}}{4}$. It is straightforward to show that the matrices $M^{(1,2)}$ commute if and only if $\mathrm{tr}(M^{(1)}) = \mathrm{tr}(M^{(2)}) = \rho + \rho^{-1} = \pm 2$ where

$$\mathrm{tr}(M^{(1,2)}) = 2\sqrt{2}\cos\left[\frac{\pi}{4}\sqrt{1+8a}\right]. \tag{6.60}$$

Therefore, we conclude that if $\mathrm{tr}(M^{(1,2)}) > 2$, one the monodromy matrices is non-resonant and the two matrices do not commute. Therefore, there is no additional first integral if $\mathrm{tr}(M^{(1,2)}) > 2$. This occurs for the following values of a:

$$a < 0,\ 1 < a < 3,\ 6 < a < 10,\ 15 < a < 21, \ldots \tag{6.61}$$

Now, the same analysis can be performed around the second straight line solution $q_1 = q_2$. In this case, the expression for the traces of the monodromy matrices is different and leads to the following conditions on the parameter a for which no second analytic first integral exists:

$$a > 3,\ 1 > a > 0,\ -\frac{1}{7} > a > -\frac{7}{11}, \ldots \tag{6.62}$$

The two sets of conditions (6.61-6.62), taken together, lead to the following result.

Proposition 6.1 *The only values of $a \in \mathbb{R}$ for which the Hamiltonian $H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}(q_1^4 + q_2^4) + \frac{a}{2} q_1^2 q_2^2$ has a second analytic first integral, are $a = 0,\ a = 1,$ and $a = 3$.*

■

6.4.1 Hamiltonian systems with two degrees of freedom

Consider a two-degree-of-freedom Hamiltonian system, $H = H(q_1, q_2, p_1, p_2)$, with a canonical Poisson bracket and assume that there exists a straight line solution

$$a_1 q_1(t) + a_2 q_2(t) = 0. \tag{6.63}$$

Without loss of generality, we can introduce a rotation of the variables q_1, q_2 which leads to $q_2 = 0$. Furthermore, we assume that in these new variables $p_2 = 0$ (this occurs if the Hamiltonian has a diagonal kinetic part) and $(q_1(t), p_1(t))$ is not an equilibrium point. We can linearize the equations of motion around the solution $\hat{\mathbf{x}} = (q_1(t), 0, p_1(t), 0)$ to obtain the normal variational equation

$$\dot{\mathbf{u}} = \begin{bmatrix} \partial^2_{q_1 p_1} H(\hat{\mathbf{x}}) & \partial^2_{p_1^2} H(\hat{\mathbf{x}}) \\ -\partial^2_{q_1^2} H(\hat{\mathbf{x}}) & -\partial^2_{p_1 q_1} H(\hat{\mathbf{x}}) \end{bmatrix} \mathbf{u}. \tag{6.64}$$

We now analyze the variational equation. The following lemma relates the existence of a first integral for the variational equation to the existence of second first integral for the Hamiltonian system (Ziglin, 1983a) (see also Yoshida (1987a) for a proof).

Lemma 6.1 (Ziglin's lemma). *If the two-degree-of-freedom Hamiltonian system $H = H(\mathbf{x})$ has a second analytic first integral, then the linear system (6.64) has a homogenous first integral of the form*

$$J_k = \sum_{i=0}^{k} c_i(\hat{\mathbf{x}}(t)) u_1^i u_2^{k-i}. \tag{6.65}$$

We can find conditions for the existence of a first integral for the variational equation by evaluating J_k on the closed circuit on $\Gamma = \{q_1(t), p_1(t)\}$, the Riemann surface of the solution with local coordinate $t \in \mathbb{C}$. A closed circuit $\gamma \subset \Gamma$ is obtained by following a path in the complex t-plane. For instance, if $\hat{\mathbf{x}}(t)$ is doubly periodic, two different closed paths on Γ are obtained by considering two paths corresponding to the two fundamental periods of $\hat{\mathbf{x}}(t)$. In this case, Γ is a real two-dimensional torus. Let $\Phi(t)$ be a fundamental solution matrix. If we evaluate $\Phi(t)$ starting at $t = t_0$ along a closed path γ, starting at t_0 and ending at t_1, we obtain a new fundamental solution $\Phi(t_1)$ related to $\Phi(t_0)$ by

$$\Phi(t_1) = M([\gamma])\Phi(t_0), \tag{6.66}$$

where $M([\gamma])$ is a *monodromy matrix* that depends only on the homotopy class of γ (see Section 3.1). The set of all such monodromy matrices for a fixed t_0 forms a group structure, the so-called *monodromy group* G of the equation (6.64).[3] Due to the Hamiltonian nature of original system, the monodromy matrices are area-preserving ($\det(M) = 1$), that is, the monodromy group is a subgroup of $\mathrm{SL}(2,\mathbb{C})$, the set of 2×2 matrices with determinant ± 1. We can now state a simplified version of Ziglin's theorem (Ziglin, 1983a).

Theorem 6.5 (Ziglin) *If the two-degree-of-freedom Hamiltonian system $H = H(\mathbf{x})$ has a second analytic first integral in the neighborhood of the solution $\hat{\mathbf{x}}(t)$ and one of the monodromy matrices, say $M^{(1)}$, is non-resonant, then any other matrix $M^{(2)} \in G$, either (i) commutes with $M^{(1)}$, or (ii) is traceless.*

Proof. The first integral J_k is invariant under the action of the monodromy group since $J_k(\mathbf{u}, t_0) = J_k(M\mathbf{u}, t_1)$ for any matrix M in the monodromy group. But, since the coefficients of J_k depend only on t through $\hat{\mathbf{x}}(t)$ we have

$$J_k(\mathbf{u}, t_0) = J_k(M\mathbf{u}, t_0)\ \forall\ M \in G. \tag{6.67}$$

Let $M^{(1)}$ be a non-resonant monodromy matrix with eigenvalues ρ, ρ^{-1}, that is, neither eigenvalues is a root of the identity, and consider the first integral J_k in the variables (ξ, η) where $M^{(1)}$ is diagonal. In these variables, the relation $J_k(\mathbf{u}, t_0) = J_k(M\mathbf{u}, t_0)$ reads

$$\sum_{i=0}^{k} d_i(\hat{\mathbf{x}}(t_0))\xi(t_0)^i\eta(t_0)^{k-i} = \sum_{i=0}^{k} d_i(\hat{\mathbf{x}}(t_0))\rho^{i-k}\xi(t_0)^i\eta(t_0)^{k-i}, \tag{6.68}$$

from which we conclude that $i = k$ and $J_k = d(\hat{\mathbf{x}}(t_0))(\xi\eta)^{k/2}$. In the same basis where $M^{(1)}$ is diagonal,

$$M^{(2)} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \tag{6.69}$$

with $\det(M^{(2)}) = \alpha\delta - \beta\gamma$ and, following the previous reasoning (see Equation (6.56)), we conclude that either

1. $\beta = \gamma = 0$ (*i.e.*, $M^{(1)}$ and $M^{(2)}$ commute), or
2. $\alpha = \delta = 0$, (*i.e.*, $\mathrm{tr}(M^{(2)}) = 0$).

Note that, in the second case, the effect of $M^{(2)}$ is to permute the eigenvectors of $M^{(1)}$, that is, if $\mathbf{v}^{(i)}$ are the eigenvectors of $M^{(1)}$, we have $M^{(2)}\mathbf{v}^{(1)} = \beta\mathbf{v}^{(2)}$ and $M^{(2)}\mathbf{v}^{(2)} = \gamma\mathbf{v}^{(1)}$. □

If $M^{(1)}$ is not semi-simple, its eigenvalues are 1 or -1 and following the basic reasoning of the proof, Ziglin's theorem can be generalized (Yoshida *et al.*, 1987a; Yoshida *et al.*, 1987b) (The proof is left as an exercise).

Theorem 6.6 *If the two-degree-of-freedom Hamiltonian system $H = H(\mathbf{x})$ has a second analytic first integral in the neighborhood of the solution $\hat{\mathbf{x}}(t)$ and one of the monodromy matrices $M^{(1)}$ is not semi-simple, then in the basis where $M^{(1)}$ is lower triangular, any other matrix $M^{(2)} \in G$ is also lower triangular with resonant eigenvalues.*

However, in practical applications the computation of the monodromy group can not be performed algorithmically. Nevertheless, there are many interesting cases where general statements can be obtained. In general, when an exact form of the monodromy group cannot be obtained, the traces of the monodromy matrices can be computed either numerically (Grammaticos *et al.*, 1987) or by perturbation (Vivolo, 1997).

[3] The monodromy group G depends on the base point t_0. However, different choices of t_0 lead to isomorphic groups.

Homogeneous potentials

We now study the case of two-degree-of-freedom Hamiltonians with a diagonal kinetic part and homogeneous potentials (Yoshida, 1987a). That is, Hamiltonians of the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{6.70}$$

where $V(\epsilon q_1, \epsilon q_2) = \epsilon^k V(q_1, q_2)$ is of degree k. Let $\mathbf{c} = (c_1, c_2)$ be the solution of $c_i = \frac{\partial V}{\partial q_i}(\mathbf{c}),\ i = 1, 2$. We define both the *integrability coefficient* λ,

$$\lambda = \frac{\partial^2 V}{\partial q_1^2}(\mathbf{c}) + \frac{\partial^2 V}{\partial q_2^2}(\mathbf{c}) - k + 1, \tag{6.71}$$

and the regions S_k (for $k \neq 0, \pm 2$):

1. $k \geq 3$
   $S_k = \{\lambda < 0, 1 < \lambda < k-1, \dots, j(j-1)k/2 + j < \lambda < j(j+1)k/2 - j, \dots\}$;
2. $S_1 = \mathbb{R} - \{0, 1, 3, 6, \dots, j(j+1)/2, \dots\}$;
3. $S_{-1} = \mathbb{R} - \{1, 0, -2, -5, \dots, -j(j+1)/2, \dots\}$;
4. $k \leq -3$
   $S_k = \{\lambda > 1, 0 > \lambda > -|k| + 2, -|k| - 1 > \lambda > -3|k| + 3,$
   $\dots, -j(j-1)|k|/2 - j + 1 > \lambda > -j(j+1)|k|/2 - j, \dots\}$.

Theorem 6.7 (Yoshida) *Let λ be the integrability coefficient of the Hamiltonian $H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2)$ where V is a potential of degree k with $k \neq 0, \pm 2$. If λ lies in the region S_k, then no additional complex analytic first integral for H exists.*

The main idea of the proof follows the steps of Example 6.6 and consider the variational equations around a particular solution, $\varphi = \varphi(t)$, of $\ddot{\varphi} + \varphi^{k-1} = 0$. The variational equation around this solution can be transformed into the Gauss hypergeometric equation by a change of independent variable $z = \varphi^k$. The monodromy group of the variational equation can be computed from the known explicit form of the monodromy matrices of the hypergeometric equation. Let M be in the monodromy group. Then, the traces of the monodromy matrices can be obtained and again the condition $|\mathrm{tr}(M)| > 2$ provides us with an explicit condition on the parameter λ.

As an example, we prove a result previously stated in Chapter 4, the nonintegrability of the soft hyperbolic billiard (Icthiaroglou, 1989).

Example 6.7 The hyperbolic billiard. Consider the soft hyperbolic billiard (4.143)

$$H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\gamma}(q_1^2 q_2^2)^\gamma. \tag{6.72}$$

We use Yoshida's theorem to prove the nonintegrability of the system. First, we compute (c_1, c_2) from the system

$$-2c_1^{2\gamma} c_2^{2\gamma} = c_1^2, \tag{6.73.a}$$
$$-2c_1^{2\gamma} c_2^{2\gamma} = c_2^2. \tag{6.73.b}$$

That is, $c_1 = \pm c_2$ and $c_1^{4\gamma - 2} = -1/2$. The integrability coefficient is

$$\lambda = \frac{\partial^2 V}{\partial q_1^2}(\mathbf{c}) + \frac{\partial^2 V}{\partial q_2^2}(\mathbf{c}) - (k-1) = -1, \tag{6.74}$$

where $k = 4\gamma$. We conclude that $\lambda \in S_4$. That is, for $2\gamma \in \mathbb{Z} \setminus \{-2, 0, +2\}$, there is no other analytical first integral than H itself. ■

6.4.2 Ziglin's theorem in n dimensions

Ziglin's theorem was originally formulated for n-degree-of-freedom Hamiltonians. A $n \times n$ matrix with eigenvalues $\boldsymbol{\rho} = (\rho_1, \dots, rho_n)$ is *non-resonant* if $\boldsymbol{\rho}^{\mathbf{i}} = 1$ with $\mathbf{i} \in \mathbb{Z}^n$ implies $\mathbf{i} = \mathbf{0}$.

Theorem 6.8 (Ziglin) *If an n-degree-of-freedom Hamiltonian system $H = H(\mathbf{x})$ has n analytic first integrals in the neighborhood of the solution $\hat{\mathbf{x}}(t)$ and if one of the monodromy matrices $M^{(1)}$ is non-resonant, then any other matrix $M^{(2)} \in G$ either (i) commutes with $M^{(1)}$, or (ii) permutes the eigenspaces of $M^{(1)}$.*

Explicit results relating the monodromy group with Kovalevskaya exponents can be obtained in the particular case of Hamiltonians with a diagonal kinetic contribution and an homogeneous potential of the form

$$H = \frac{1}{2}(p_1^2 + \dots + p_n^2) + V(q_1, \dots, q_n), \tag{6.75}$$

where $V(\mathbf{x})$ is homogeneous of degree k where $k \neq 0, \pm 2$. As already stated in Chapter 5, the Kovalevskaya exponents always come by pairs $\rho_i + \rho_{i+n} = \frac{k+2}{k-2}$ and we can define the difference between two exponents of each pair as $\Delta\rho_i = \rho_{i+n} - \rho_i$.

Theorem 6.9 (Yoshida, 1989) *If the n numbers $\Delta\rho_i$ are $\mathbb{Q}$-independent, then the Hamiltonian (6.75) has no additional first integral besides H.*

Proof. To compute the Ziglin conditions for nonintegrability, we consider the straight line solution

$$\hat{\mathbf{q}}(t) = \boldsymbol{\beta}\varphi(t), \tag{6.76}$$

where $\varphi(t)$ is a solution of $\ddot{\varphi} + \varphi^{k-1} = 0$ and $\boldsymbol{\beta}$ is a solution of $\partial_{\mathbf{x}} V(\boldsymbol{\beta}) = \boldsymbol{\beta}$. The coefficients $\boldsymbol{\beta}$ are related to the constant coefficients of the scale invariant solution $\mathbf{q} = \boldsymbol{\alpha}\tau^p$ by

$$\boldsymbol{\alpha} = [-p(p+1)]^{\frac{p}{2}} \boldsymbol{\beta}, \tag{6.77}$$

with $p = 2/(k-1)$. The linear variational equation around $\hat{\mathbf{q}}(t)$ reads

$$\ddot{\mathbf{u}} = -\varphi^{k-1}\partial^2_{\mathbf{q}} V(\boldsymbol{\beta})\mathbf{u}, \tag{6.78}$$

where $\partial^2_{\mathbf{q}} V(\boldsymbol{\beta})$ is the Hessian matrix of the potential evaluated on $\boldsymbol{\beta}$. We now choose a system of coordinates $\boldsymbol{\xi}$ where $\partial^2_{\mathbf{q}} V(\boldsymbol{\beta})$ is diagonal. That is,

$$\ddot{\xi}_i = -\lambda_i \varphi^{k-2}\xi_i, \qquad i = 1, \dots, n. \tag{6.79}$$

where $(\lambda_1, \dots, \lambda_n) = \mathrm{Spec}(\partial^2_{\mathbf{q}} V(\boldsymbol{\beta}))$ and $\lambda_n = k-1$. The normal variational equations are obtained by considering the $(n-1)$ first equations of (6.79). Since the normal variational equations form a system of $(n-1)$ decoupled equations, the monodromy matrix is block-diagonal. That is,

$$M = \mathrm{diag}(M(\lambda_1), \dots, M(\lambda_{n-1})), \tag{6.80}$$

where every matrix $M(\lambda_i)$ is a 2×2 matrix with determinant unity. We now look for two non-resonant monodromy matrices $M^{(1,2)}$ corresponding to two different paths $C_{1,2}$ such that none of the blocks $M^{(1,2)}(\lambda_i)$ commute. According to Ziglin's theory, the existence of such matrices implies the non-existence of an additional first integral. These matrices can be obtained by (i) mapping the system of normal variational equations to a system of Gauss hypergeometric equations by the change of independent variable $z = \varphi^k$; and (ii) considering the monodromy matrices associated with the pre-images of the paths $(\gamma_0\gamma_1)^{4k}$ and $(\gamma_1\gamma_0)^{4k}$ in the z-plane, where, as before, γ_0 and γ_1 are the closed paths around $z = 0$ and $z = 1$, respectively. The matrices $M^{(1,2)}$ associated with such paths are of the form $M^{(1,2)} = \mathrm{diag}(M^{(1,2)}(\lambda_1), \dots, M^{(1,2)}(\lambda_{n-1}))$ and have the following properties.

1. The blocks $M^{(1,2)}(\lambda_i)$ share the same trace:

$$\operatorname{tr}(M^{(1,2)}(\lambda_i)) = 2\cos(2\pi\theta_i), \qquad i = 1, \dots, n-1, \tag{6.81}$$

   where $\theta_i = \sqrt{(k-2)^2 + 8k\lambda_i}$ and $\theta_n = 3k - 2$.

2. The blocks $M^{(1,2)}(\lambda_i)$ commute only when $\operatorname{tr}(M^{(1,2)}(\lambda_i)) = \pm 2$ or $k = \pm 2$.

The non-resonance condition is obtained from the eigenvalues $\sigma_j = e^{2\pi i\theta_j}$ of the matrices $M^{(1,2)}(\lambda_i)$. We find that, if the n numbers $\theta_1, \dots, \theta_n$ are rationally independent, the matrices $M^{(1,2)}$ are non-resonant and none of the pairs $M^{(1,2)}(\lambda_i)$ commute unless $k = \pm 2$.

Finally, the difference between a pair of Kovalevskaya exponents ρ_i, ρ_{i+n} such that $\rho_i + \rho_{i+n} = 2g + 1$ can be computed in terms of the eigenvalues θ_i and it is found that $\Delta\rho_i = |k-2|\theta_i$. That is,

$$\Delta\rho_i = \sqrt{1 + \frac{8k\lambda_i}{(k-2)^2}}, \tag{6.82}$$

which completes the proof. □

Example 6.8 Reduce Yang-Mills equations. Consider the Yang-Mills quadratic potential in three dimensions

$$V = q_1^2 q_2^2 + q_2^2 q_3^2 + q_1^2 q_3^2. \tag{6.83}$$

There is a straight-line solution with $\boldsymbol{\beta} = (\sqrt{2}/2, \sqrt{2}/2, 0)$ and $\boldsymbol{\lambda} = (2, -1, 3)$. The difference of Kovalevskaya exponents is simply given by (6.82), that is, $(\Delta\rho_1, \Delta\rho_2, \Delta\rho_3) = (\sqrt{17}, i\sqrt{7}, 5)$. These numbers are clearly rationally independent and we conclude that system (6.75) with potential (6.83) has no additional analytic first integral. ■

6.4.3 More on Ziglin's theory

Ziglin's theory has been successfully applied to prove the nonintegrability of the following systems.

1. Hamiltonians with n degrees of freedom and global and symmetric coupling (Umeno, 1995);
2. three-degree-of-freedom Hamiltonian systems (Churchill *et al.*, 1995);
3. some generalized Toda lattices (Yoshida *et al.*, 1987a; Yoshida, 1988);
4. the planar three-body problem in a neighborhood of the Lagrangian solution (Tsygvintsev, 2000);
5. a class of perturbed Kepler problems (Yoshida, 1987b);
6. the Störmer problem (Almeida *et al.*, 1992);
7. the Calogero-Moser Hamiltonian with quartic potentials (Françoise & Irigoyen, 1990; Françoise & Irigoyen, 1993);
8. the classical Zeeman Hamiltonians (Kummer & Sáenz, 1994);
9. some generalization of rigid body motion (Christov, 1994);
10. some generalized Hénon-Heiles potentials (Ito, 1985; Ito, 1987; Vivolo, 1997);
11. the ABC-flow (Ziglin, 1996; Ziglin, 1998);
12. some perturbed separable planar Hamiltonians (Meletlidou, 2000).

6.4.4 The Morales-Ruiz and Ramis theorem

A major improvement on Ziglin's theory was proposed by Morales-Ruiz and Ramis (1998; 1999). It is based on differential Galois theory (Picard-Vessiot theory) (Singer, 1989) and relates the integrability of a Hamiltonian system to the *solvability* of its variational equation around a particular solution (Morales-Ruiz & Simo, 1994; Morales-Ruiz & Simo, 1996). Roughly speaking, the variational equation is *solvable*, in the sense of differential Galois theory, if the solution can be obtained by combination of quadratures, exponential of quadratures and algebraic functions. The advantage of this approach is that general results on the solvability of linear equations are available. For instance, the solvability of a second order algebraic linear equation can be tested by using Kovacic's algorithm (Kovacic, 1986; Ulmer & Weil, 1996; Rod & Sleeman, 1995). This algorithm gives a fundamental set of solutions in closed form when the equation is solvable and when the algorithm fails the equation is non-solvable. Another major result on the solvability of linear equations is Kimura's theorem which gives necessary and sufficient conditions for the solvability of Gauss' hypergeometric equation (Kimura, 1969). As an application, Morales-Ruiz and Ramis (1999) generalized Theorem 6.7 and obtained stronger conditions on the integrability coefficient λ defined by (6.71). They also proved the nonintegrability in the Liouville sense of the Bianchi IX cosmological model (Latifi *et al.*, 1994), the Sitnikov model (Martinez-Alfaro & Chriald, 1992), and other Hamiltonian systems (Morales-Ruis & Ramis, 1998). This method was applied by Sáenz (2000) to prove the nonintegrability of the Dragt-Finn model of magnetic confinement. Furthermore, Yoshida (1999) used Morales-Ruiz-Ramis' theorem to justify the weak-Painlevé conjecture and showed that for a Liouville integrable two-degree-of-freedom Hamiltonians with a kinetic diagonal part and an homogeneous potential of degree k the Kovalevskaya exponents are rational numbers except for the cases $k = 0$ and $k = \pm 2$.

A nice introduction to the subject can be found in Yoshida's paper (1999). For a detailed exposition of this theory see the excellent book by Morales-Ruiz (Morales-Ruiz, 1999).

Chapter 7:

Nearly integrable dynamical systems

" Any variable or flowing quantity can be differenced;
but, vice versa, any differential cannot be integrated."
Chambers (1727–1741)

In the previous chapters we built a theory of integrability and nonintegrability based primarily on singularity analysis. We showed the connection between the local meromorphicity of solutions and the global existence of first integrals. This is a first step in a more general theory aimed at understanding the relationship between global dynamics in phase space and the local study of singularities. We exhibited many examples of systems whose integrable features are related to non-critical singularities. However, singularity analysis is not a very flexible tool for nonintegrable systems. The Painlevé test, when applied to such systems, does not appear to provide any insight into their dynamics. It is even more puzzling to consider systems whose real time behavior is completely regular, but whose solutions introduce branch points in the complex plane. Despite their apparent simplicity, these systems cannot be studied using standard singularity analysis. The local meromorphicity of their solutions, or lack thereof, seems to be unrelated to their long-time dynamics.

In Chapter 5, nonintegrable systems were considered in the complex plane using the *Ψ-series*. These series were shown to give a picture of singularities clustering, and in some specific cases, they were shown to determine first integrals (Levine & Tabor, 1988). Despite the importance of the issues raised by the Ψ-series, the exact relation between multi-valuedness and complex behavior is not yet understood (Bountis *et al.*, 1987).

We now pursue our study by considering the effect of logarithmic critical branch points on the geometry of nonintegrable dynamical systems. Our study is based on homoclinic curves in phase space and their destruction under perturbation.

The importance of homoclinic orbits and of transverse homoclinic intersections (*i.e.,* transverse intersection of the stable and unstable manifolds of the fixed point) can be found in a result by Smale (1967), which generalizes a theorem by Birkhoff (1935). The Smale-Birkhoff theorem states that the existence of transverse homoclinic intersections for diffeomorphisms is sufficient for the existence of an invariant Cantor set in which the periodic orbits are dense. Therefore, we need a criterion for the existence of an homoclinic intersection in the Poincaré map defined by the suspended flow of the perturbed system. Melnikov (1963) gave a computable criterion for the existence of transverse homoclinic intersections in perturbed integrable planar systems by defining a function which gives an approximation of the splitting distance between stable and unstable manifolds. For a nearly integrable n-dimensional system with homoclinic structure, we can define the *Melnikov vector*, a vector of functions. The analysis of the simple zeroes of these functions allows us to compute the transverse intersections of homoclinic manifolds (Gruendler, 1985; Bruhn, 1991; Chow & Yamashita, 1992; Wiggins, 1992; Gelfreich & Sharomov, 1995; Gruendler, 1996).

In this chapter, a perturbative approach for nearly integrable nonlinear differential equations is presented. Perturbative expansions, based on the Ψ-series are formal local solutions around the singularities. The series is ordered according to the perturbation parameter ϵ. At each order in ϵ, the solution has the form of a polynomial in $\log(t - t_*)$, whose coefficients are convergent Laurent series. For integrable systems under perturbations, it is possible to show that under suitable assumptions on the existence of homoclinic solutions, the leading order behavior of these Laurent series is directly connected to the Melnikov vector. This gives a direct link between

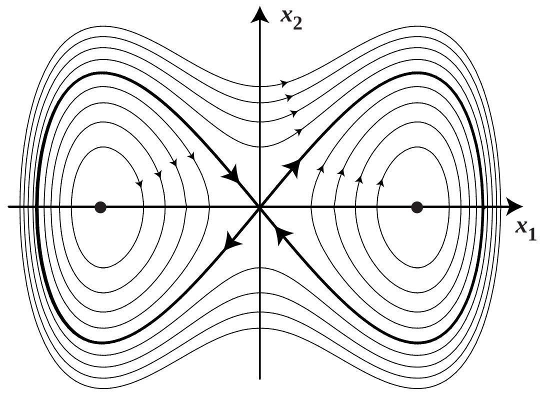

Figure 7.1: The phase portrait of the unperturbed Duffing equation ($\epsilon = 0$).

transverse intersections and the existence of critical points. Moreover, a explicit way to compute the Melnikov vector in n dimensions is proposed based on the compatibility conditions for the formal existence of the Laurent expansion for nonintegrable systems. It also provides a different geometric interpretation of the Painlevé property in terms of homoclinic structures. This approach establishes the connection between singularity structure and nonintegrable behaviors such as chaotic motion (Goriely & Tabor, 1994; Goriely & Tabor, 1995). The relationship between splitting of invariant manifolds, branching of solutions and nonintegrability has been established in Hamiltonian dynamics by many authors (Ziglin, 1980; Ziglin, 1983a; Mikolaevskii & Schur, 1983; Bountis *et al.*, 1987; Koiller, 1991; Dovbysh, 1992; Icthiaroglou, 1996; Kozlov, 1998).

In this chapter, we first give the general assumption on the unperturbed and perturbed systems. Next, we introduce a perturbative expansion around singularities. The Melnikov vector in n dimensions is then defined and computed by the method of residues. Finally, we show that the Melnikov vector can be estimated by the local perturbative expansion.

7.0.1 An introductory example

We present an outline of our results using a very simple example, the Duffing system, for which both dynamical systems analysis (Guckenheimer & Holmes, 1983; Kapitaniak, 1990; Ryabov & Vavriv, 1991; Belogortsev, 1992) and singularity analysis (Fournier *et al.*, 1988; Bountis *et al.*, 1993) have been performed in detail (see Examples 3.12 and 3.22). Following the layout of this chapter, the results for the planar conservative case are given without proofs and are illustrated on the Duffing system.

Example 7.1 The Duffing equation. The Duffing system, written as a first order, system reads

$$\dot{x}_1 = x_2, \tag{7.1.a}$$
$$\dot{x}_2 = x_1 - x_1^3 + \epsilon\cos(t). \tag{7.1.b}$$

For $\epsilon = 0$, this system is completely integrable with first integral $I = x_1^2 - x_1^2 + \frac{x_1^4}{2}$. The phase portrait of this system is shown in Figure 7.1. In particular, the phase space has a pair of homoclinic[1] curves $\hat{\mathbf{x}}(t)$ to the origin given by

$$(x_1, x_2) = (\pm\sqrt{2}\mathrm{sech}(t), \mp\sqrt{2}\mathrm{sech}(t)\tanh(t)). \tag{7.2}$$

The interior of these orbits are filled with open sets of periodic orbits. As periods increase, the curves approach the homoclinic orbit.

[1]A *homoclinic orbit* to a fixed point $\bar{\mathbf{x}}$ is an orbit $\hat{\mathbf{x}}(t)$ such that $\hat{\mathbf{x}}(t) \to \bar{\mathbf{x}}$ as $t \to \pm\infty$.

To understand the dynamic for the system for small ϵ, we consider the perturbation of the homoclinic orbits. For $\epsilon = 0$, these orbits are the union of the stable and unstable manifolds $W^{s,u}(\mathbf{0})$ to the origin. The distance between the stable and unstable manifolds can be computed to first order in the perturbation parameter ϵ in the following manner. First, we introduce the Melnikov integral for a planar conservative system. Consider an integrable system under a perturbation $\epsilon\mathbf{g}(\mathbf{x}, t)$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, t), \tag{7.3}$$

where $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{g}(\mathbf{x}, t)$ is 2π-periodic in t. For a planar conservative system with an homoclinic orbit $\hat{\mathbf{x}}(t)$ to

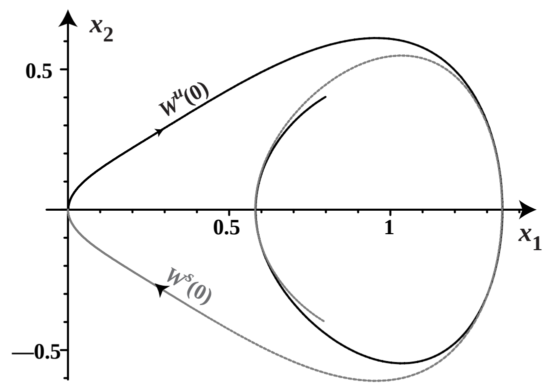

Figure 7.2: Splitting of the homoclinic orbit ($\epsilon = 0.1$).

the hyperbolic fixed point, the *Melnikov integral*, M, measures the first order approximation of the splitting distance between the stable and unstable manifolds to the fixed point (Wiggins, 1988; Perko, 1996)

$$M(t_0) = \int_{-\infty}^{+\infty} \mathbf{f}(\hat{\mathbf{x}}(t)) \wedge \mathbf{g}(\hat{\mathbf{x}}(t), t + t_0) dt, \tag{7.4}$$

where $\mathbf{f} \wedge \mathbf{g} = (f_1 g_2 - f_2 g_1)$. The arbitrary time t_0 provides us with a parametrization of the unperturbed homoclinic orbit and is considered as a reference orbit for the distance estimate. Therefore, for the Duffing system, we have

$$M(t_0) = \int_{-\infty}^{+\infty} \sqrt{2}\, \text{sech}(t) \cos(t + t_0) dt. \tag{7.5}$$

As we are confronted with an improper integral in (7.5), we take advantage of the properties of the homoclinic orbit in the complex plane. The homoclinic orbits are hyperbolic functions whose *fundamental period* in the imaginary direction is $\omega = 2\pi$ (*i.e.*, $\hat{\mathbf{x}}(t + \omega i) = \hat{\mathbf{x}}(t)\ \forall t \in \mathbb{R}$)). The strip $S = \{t \in \mathbb{C}; 0 \leq \Im(t) \leq 2\pi\}$ is the *fundamental domain* of the homoclinic solution with singularities along the imaginary axis: $t_*^{(1)} = i\pi/2$, $t_*^{(2)} = 3i\pi/2$. Therefore, we use this periodicity property to compute a contour integral on the border of the strip S which reduces to a sum of residues. More generally, the computation of any Melnikov integral can be made systematic if we use this same procedure. In Section 7.4, we will show that for systems possessing homoclinic orbits periodic in the imaginary direction, the Melnikov vector can be computed by the method of residues to give

$$M(t_0) = 2\pi i \sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} \frac{e^{ikt_0}}{(1 - e^{-\omega k})} \sum_{t_* \in S} \underset{t_*}{\text{res}}(\mathbf{f} \wedge \mathbf{g}^{(k)}(\hat{\mathbf{x}}))e^{ikt_*}. \tag{7.6}$$

This sum is taken over all of the singularities of $\hat{\mathbf{x}}(t)$ in a fundamental domain S and $\mathbf{g}^{(k)}(\hat{\mathbf{x}})$ is the k-th component of the Fourier expansion of $\mathbf{g}$ with respect to t,

$$\mathbf{g}(\mathbf{x}, t) = \sum_{k=-\infty}^{+\infty} \mathbf{g}^{(k)}(\mathbf{x}) e^{ikt}. \tag{7.7}$$

A similar form of the Melnikov integral for planar Hamiltonian system has been obtained by Ziglin (1982).

The computation of the Melnikov integral now reduces to the computation of residues at the singularities. These residues can be computed from the Ψ-series which is obtained by singularity analysis. More specifically, in Section 7.2.3, we will show that the solution of a nearly integrable system can be expanded locally around the singularity t_* of the unperturbed system:

$$\mathbf{x} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \mathbf{s}^{(ij)} \epsilon^i (\log(t - t_*))^j, \tag{7.8}$$

where $\mathbf{s}^{(ij)}$ are the convergent Laurent series around t_* which are determined recursively. The first term $\mathbf{s}_{00}$ is the Laurent series solution of the unperturbed problem.

For the Duffing system, a Ψ-series similar to the one found in Example 3.12 can be expanded to first order in ϵ. Therefore, the solution of the nonintegrable Duffing (7.1) system can be written explicitly as

$$\begin{aligned} \mathbf{x} = (t - t_*)^{\mathbf{p}} \Big\{ & \sum_{i=0}^{\infty} \mathbf{a}_i (t - t_*)^i \\ + \quad & \epsilon \Big(\sum_{i=0}^{\infty} \mathbf{b}_i (t - t_*)^i + \log(t - t_*) \sum_{i=0}^{\infty} \mathbf{c}_i (t - t_*)^{i+4} \Big) \\ + \quad & O(\epsilon^2) \Big\}, \end{aligned} \tag{7.9.a}$$

with $\mathbf{p} = (-1, -2)$ and $\mathbf{a}_0 = (\pm i\sqrt{2}, \mp i\sqrt{2})$. The coefficient $\mathbf{a}_4$ is arbitrary. To ensure this arbitrariness, we set the coefficient $\mathbf{c}_0$ to

$$\mathbf{c}_0 = \sin(t_*)(1/5, 3/5). \tag{7.10}$$

Furthermore, we will show in Section 7.4 that this choice of $\mathbf{c}_0$ is related to the residue of $(\mathbf{f} \wedge \mathbf{g})$ around t_*.

The Melnikov integral can be evaluated by the method of residues. From singularity analysis, the residues are determined by the coefficients of the Ψ-ϵ expansion. Consider expansion (7.8) and define C to be

$$\begin{aligned} C(t_*) &= \bar{\boldsymbol{\beta}}.\mathbf{c}_0(t_*) \\ &= \frac{1}{\sqrt{5}} \sin(t_*), \end{aligned} \tag{7.11}$$

where $\bar{\boldsymbol{\beta}} = (2/\sqrt{5}, 1/\sqrt{5})$ is the unit left eigenvector of the Kovalevskaya matrix (3.94) of eigenvalue $\rho = 4$ and $\mathbf{c}_0$ is given by (7.10).

Due to the periodicity of $\mathbf{g}(\mathbf{x}, t)$, C is a periodic function of t_* ($C(t_* + 2\pi) = C(t_*)$). This last quantity is proportional to the residue of $(\mathbf{f} \wedge \mathbf{g})$ at the singularity t_*. Therefore using (7.6) and (7.11), the Melnikov integral is found to be

$$M(t_0) = 2\pi i \sum_k \frac{e^{ikt_0}}{1 - e^{-\omega k}} \sum_{t_* \in S} C^{(k)}(t_*) e^{ikt_*}, \tag{7.12}$$

where $C^{(k)}(t_*)$ is the k-th component of the Fourier expansion of $C(t_*)$. For this particular case, the Melnikov integral can be readily computed:

$$\begin{aligned} M(t_0) &= \frac{\pi i}{\sqrt{5}}\left\{ e^{it_0}\frac{e^{-\pi/2}-e^{-3\pi/2}}{1-e^{-2\pi}} - e^{-it_0}\frac{e^{\pi/2}-e^{3\pi/2}}{1-e^{2\pi}}\right\} \\ &= \frac{\pi}{\sqrt{5}}\text{sech}(\pi/2)\sin(\text{t}_0). \end{aligned} \tag{7.13}$$

Therefore, the Melnikov vector can be computed by singularity analysis for nonintegrable systems and a direct relationship between the existence of logarithmic branch points and homoclinic intersections can be established. ■

7.1 General Setup

We consider systems of n first order ordinary differential equations in $\mathbb{R}^n$,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, \tag{7.14}$$

where $\mathbf{f}$ and $\mathbf{g}$ are analytic functions. The system obtained by setting $\epsilon = 0$ is referred to as the *unperturbed system*. We are interested in the behavior of the unperturbed system under small perturbations $0 < \epsilon \ll 1$. More specifically, we study the perturbation of a homoclinic manifold under a periodic forcing which can be either external or internal. In the first case, the unperturbed system is autonomous and the forcing is explicitly described by a time-dependent function (Holmes, 1980). In the second case, the forcing is produced by the coupling of two subsystems, one of which behaves periodically (Holmes, 1982).

7.1.1 General assumptions

• **Assumption 1** (*Integrability*). The unperturbed system is completely integrable. Complete integrability is defined as algebraic complete integrability for non-Hamiltonian systems (see Definition 2.19) and Liouville integrability for Hamiltonian systems (see Theorem 6.2).
• **Assumption 2** (*Hyperbolicity of the fixed point*). The unperturbed system has a hyperbolic fixed point at $\mathbf{x}_0 = \mathbf{0}$, with linear eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ with $\Re(\lambda_i) \neq 0\ \forall\ i$ and a semi-simple Jacobian matrix $D\mathbf{f}(\mathbf{0})$.
• **Assumption 3** (*Periodicity of the homoclinic solution*). There exists a homoclinic curve $\hat{\mathbf{x}}(t)$, a solution of the unperturbed problem connecting the hyperbolic fixed point to itself with the property that there exists $\omega \in \mathbb{R}_0^+$ such that

$$\hat{\mathbf{x}}(t + \omega i) = \hat{\mathbf{x}}(t) \quad \forall\ t \in \mathbb{R}. \tag{7.15}$$

• **Assumption 4** (*Analytic continuation*). The homoclinic solution of the unperturbed system can be analytically continued in a strip, S, where

$$S = \{t \in \mathbb{C};\ 0 \le \Im(t) \le \omega\}. \tag{7.16}$$

Moreover, the homoclinic solution has a finite number of poles N in strip S, around which, the homoclinic solution can be expanded in a Laurent series

$$\mathbf{x} = (t - t_*)^{\mathbf{p}} \sum_{j=0}^{\infty} \mathbf{a}_j (t - t_*)^j, \tag{7.17}$$

with $\mathbf{a}_j \in \mathbb{C}^n\ \forall\ j;\ \mathbf{p} \in \mathbb{Z}^n$.
• **Assumption 5** (*Periodicity of the perturbation*). The perturbation $\mathbf{g}(\mathbf{x}, t)$ is periodic in t and polynomial in $\mathbf{x}$. That is,

$$\mathbf{g}(\mathbf{x}, t + 2\pi) = \mathbf{g}(\mathbf{x}, t) \quad \forall\ t \in \mathbb{R}. \tag{7.18}$$

• **Assumption 6** (*Non dominant perturbations*). At a singularity $t = t_*$ of the homoclinic orbit, $\mathbf{g}(\hat{\mathbf{x}}(t), t)$ is *non-dominant*, that is, there exists a vector $\mathbf{p}_g \in \mathbb{N}_0^n, \boldsymbol{\delta} \in \mathbb{C}_0^n$, such that

$$\mathbf{g}(\hat{\mathbf{x}}(t), t) \underset{t \to t_*}{\sim} \boldsymbol{\delta}(t - t_*)^{\mathbf{p}+\mathbf{p}_g-1}. \tag{7.19}$$

7.1.2 Comments on the assumptions

• **A1**. The starting point of the analysis of nearly integrable systems is a completely integrable system. In our case, we start with an integrable n-dimensional system with k first integrals, where $k = (n-1)$ for general systems or $k = l/2$ for Hamiltonian systems. In general, for an n-dimensional system, $(n-1)$ first integrals are required. For simplicity, the first integrals are not time-dependent in this study.
• **A2**. This assumption guarantees the hyperbolicity of the fixed point and is a local condition for the existence of homoclinic curves. The semi-simple assumption on the Jacobian matrix is a technical assumption used to compute Floquet exponents and to control the behavior of the homoclinic curves as $t \to \pm\infty$.
• **A3**. The periodicity of the homoclinic curve in the imaginary direction is a central assumption to apply the method of residues in the computation of the Melnikov integrals. For planar Hamiltonian systems, this property can be proved exactly and the period ω is given by $\omega = \Re(\frac{2\pi}{|\lambda|})$, where λ is the linear eigenvalue of the hyperbolic fixed point (Ziglin, 1982). Whether or not this property holds generally in higher dimensions is still an open question. It is possible that further restrictions on the unperturbed system will be required to prove this assumption but it seems to hold in most examples (Bertozzi, 1988; Gruendler, 1985; Koch, 1986; Wiggins, 1988; Bruhn, 1989; Bruhn, 1991). This assumption can be checked either analytically, when the explicit solution is known, or numerically by path-integration in the complex plane (Chang & Corliss, 1980). The theory presented in this chapter can also be generalized to the case of heteroclinic connections between two hyperbolic fixed points. However, for clarity, we only present the case of homoclinic intersections.
• **A4**. The analytic continuation in the fundamental domain of the homoclinic solution is necessary for the estimation of the Melnikov integral by the method of residues. The sum over the residues is finite since the number of poles in the strip S is finite. The main assumption about the homoclinic solution is that it can be locally expanded in Laurent series around the singularity which allows us to find a local expansion of the solution of the perturbed system.
• **A5**. The periodicity of the perturbation allows us to consider an autonomous version of the perturbed system (7.14) evolving on $\mathbb{R}^n \times S^1$. That is,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, \theta), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{7.20.a}$$
$$\dot{\theta} = 1, \qquad \theta \in S^1. \tag{7.20.b}$$

In the case of a quasi-periodic forcing, it is still possible to define a Melnikov vector (Wiggins, 1987; Haller & Wiggins, 1993). Such a situation will not be considered here.
• **A6**: This assumption allows us to consider the perturbation $\mathbf{g}(\mathbf{x}, t)$ as a regular perturbation entering at higher powers of $(t - t_*)$ in the Laurent expansion. This is the starting point of a local perturbative scheme in the complex plane. Examples of systems with dominant perturbations will also be given.

7.2 A perturbative singularity analysis

In this section, we first recall the main ingredients of singularity analysis and show that for nearly integrable systems, there exist local expansions in the perturbation parameter.

7.2.1 The Painlevé test

We briefly recall the Painlevé test as introduced in Section 3.8. First, we find a weight-homogeneous decomposition $\mathbf{f}^{(0)}$ of the vector field $\mathbf{f} = \mathbf{f}^{(0)} + \sum_i \mathbf{f}^{(i)}$ such that the *dominant* system $\dot{\mathbf{x}} = \mathbf{f}^{(0)}$ is scale-invariant. That is, the system admits a solution $\mathbf{x} = \boldsymbol{\alpha}(t - t_*)^{\mathbf{p}},\ |\boldsymbol{\alpha}| \neq 0,\ \mathbf{p} \in \mathbb{Z}^n$ with at least one negative entry. Each *balance*

$(\boldsymbol{\alpha}, \mathbf{p})$ defines a different expansion. Second, we compute the Kovalevskaya exponents which are the eigenvalues of matrix K,

$$K = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \text{diag}(\mathbf{p}). \tag{7.21}$$

The Kovalevskaya exponents are labeled $\rho_i,\ i = 1, \dots, n$ with $\rho_1{=}{-}1$. The necessary condition for the existence of a Laurent series (see Painlevé test #1 of Section 3.9.1) is that all Kovalevskaya exponents are integer valued ($\rho_i \in \mathbb{Z}$). Third, we check that there is a Laurent series solution with as many arbitrary constants as the number of positive Kovalevskaya exponents (see Painlevé test #3 of Section 3.9.3). The coefficients are computed by inserting the full Laurent series,

$$\mathbf{x} = (t - t_*)^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{i=1}^{\infty} \mathbf{a}_i (t - t_*)^i \right), \tag{7.22}$$

into the original system and by determining explicitly the recursion relation for the coefficients $\mathbf{a}_j$,

$$K\mathbf{a}_j = j\mathbf{a}_j - \mathbf{P}_j(\mathbf{a}_1, \dots, \mathbf{a}_{j-1}), \tag{7.23}$$

where $\mathbf{P}_j$ is polynomial in its variables. If $j = \rho$ is an eigenvalue of K, then in order to solve the linear system for $\mathbf{a}_r$, we need the compatibility conditions

$$C_\rho = \bar{\boldsymbol{\beta}}^{(\rho)}.\mathbf{P}_\rho = 0 \quad \forall\ \rho \geq 0, \tag{7.24}$$

where $\bar{\boldsymbol{\beta}}^{(\rho)}$ is the eigenvector of K^T of eigenvalue ρ. If condition (7.24) is fulfilled for all positive Kovalevskaya exponents ρ and for all balances $(\boldsymbol{\alpha}, \mathbf{p})$, the system passes Painlevé test #3 in the variables $(\mathbf{x}, t)$.

7.2.2 The Ψ-series

If one of the aforementioned conditions is not satisfied, then the system does not have the Painlevé property in the set of variables $(\mathbf{x}, t)$. Different cases can be distinguished. First, if some components of vector $\mathbf{p}$ or of Kovalevskaya exponents $\boldsymbol{\rho}$ are rational, then it is sometimes possible to build a variable transformations such that in a new set $(\mathbf{x}', t')$, the system satisfies the Painlevé test (as explained in Chapter 4). Second, if either the Kovalevskaya exponents are irrational, or one of the compatibility conditions is not satisfied, the solution exhibits movable critical points and the system is not Painlevé integrable.

If the compatibility conditions are not satisfied, then from Theorem 3.2, there exists a Ψ*-series* solution of the form

$$\mathbf{x}(t) = (t - t_*)^{\mathbf{p}} \left(\boldsymbol{\alpha} + \sum_{\mathbf{i},\ |\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}(\log(t - t_*))(t - t_*)^{(\boldsymbol{\rho}, \mathbf{i})} \right), \quad (\boldsymbol{\rho}, \mathbf{i}) = \sum_{j=k}^{n+1} \rho_j i_j, \tag{7.25}$$

where $\mathbf{c}_{\mathbf{i}}(\log(t - t_*))$ is polynomial in $\log(t - t_*)$, $\{\rho_1, \dots, \rho_n,\}$ are the eigenvalues of the Kovalevskaya matrix K and $\rho_{n+1} = q$ (as defined in Theorem 3.2).

Different studies have shown both that these Ψ-series contain valuable information on the clustering of singularities in the complex plane and that some subsets of the Ψ-series can be resummed (see Section 3.11). This resummation process can be achieved when the differential equations the Ψ-series obey can be explicitly solved (Levine & Tabor, 1988).

We now propose a perturbative expansion in the parameter ϵ for system (7.14). This expansion takes into account the fact that the system is nearly integrable.

7.2.3 Epsilon-expansion for the Ψ-series

Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, t)$ and assume both that the unperturbed system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ passes Painlevé test #3 and that the perturbation term $\mathbf{g}(\mathbf{x}, t)$ satisfies assumption A6. Furthermore, assume that for $\epsilon \neq 0$, this system is nonintegrable with at least one of the compatibility conditions (7.24) for the formal Laurent solutions not satisfied. The compatibility conditions for Kovalevskaya exponent ρ can be written as a polynomial in ϵ of degree ρ. That is,

$$C_\rho = \sum_{i=1}^{\rho} c_{\rho,i}\epsilon^i. \tag{7.26}$$

As a consequence, for all but finitely many ϵ, there is no Laurent solution for the perturbed system.

We show that there exist formal series around the singularities of the unperturbed system in the variables $(t-t_*)$ and $\log(t-t_*)$, ordered by the parameter ϵ.

Proposition 7.1 *Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x},t)$ and assume both that the unperturbed system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ passes Painlevé test #3 and that the perturbation term $\mathbf{g}(\mathbf{x},t)$ satisfies assumption A6. Then, for ϵ small enough, the perturbed system admits a formal ϵ expansion of the Ψ-series given by*

$$\mathbf{x} = \sum_{i=0}^{\infty}\sum_{j=0}^{i} \mathbf{s}^{(ij)}\epsilon^i Z^j, \tag{7.27}$$

where $Z = \log(t-t_*)$ and $\mathbf{s}^{(ij)}$ are convergent Laurent series in a punctured disk $B_{ij}(t_*)$.

Proof. The existence of expansion (7.27) is a direct consequence of the formal existence of the Ψ-series guaranteed by Theorem 3.2. Indeed, series (7.27) is a formal expansion of the Ψ-series (7.25) in ϵ. We show that *(i)* each coefficient $\mathbf{c_i}$ in (7.25) is polynomial in ϵ of degree less than or equal to $|\mathbf{i}| - \max\{\mathbf{p_g}\}$ (where $\mathbf{p}_g$ is the vector of exponents describing the behavior of $\mathbf{g}$ at the singularity (7.19)), and that *(ii)* the series $\mathbf{s}^{(ij)}$ is convergent.

(i) Consider $\mathbf{s}^{(ij)}$ as an independent variable. We want to show that at each order in ϵ, the solution can be locally written as expansion (7.8). To do so, we write (7.8) as

$$\mathbf{x} = \sum_{i=0}^{\infty}\mathbf{x}^{(i)}\epsilon^i, \quad \mathbf{x}^{(i)} = \sum_{j=0}^{i}\mathbf{s}^{(ij)}Z^j, \tag{7.28}$$

where $\mathbf{s}^{(ij)}$ is meromorphic around t_*. This ansatz is clearly valid to $O(\epsilon^0)$ since for $\epsilon = 0$, all local solutions around the movable singularities can be expanded in Laurent series $\mathbf{x}^{(0)}$. Now, assume by induction that if the expansion (7.28) is valid up to $O(\epsilon^{i-1})$ then it is valid to $O(\epsilon^i)$. This means that if to $O(\epsilon^{i-1})$ there exists a solution $\mathbf{x}^{(i-1)}$ of form (7.28), then there exists a solution $\mathbf{x}^{(i)}$ with polynomial dependence in Z. To $O(\epsilon^i)$, the variable $\mathbf{s}^{(ij)}$ obeys a set of i coupled systems of n linear ODEs. This set is obtained by equating the terms of the same powers in Z in the differential equation satisfied by $\mathbf{x}^{(i)}$. To obtain this set of equations for $\mathbf{s}^{(ij)}$, we first take the time derivative of $\mathbf{x}^{(i)}$ to obtain

$$\dot{\mathbf{x}}^{(i)} = \sum_{j=0}^{i-1}\left(\dot{\mathbf{s}}^{(ij)} + (j+1)(t-t_*)^{-1}\mathbf{s}^{(i,j+1)}\right)Z^j + \dot{\mathbf{s}}^{(ii)}Z^i, \tag{7.29}$$

where we have used the relation $\dot{Z} = (t-t_*)^{-1}$. Second, we compute the expansion of the perturbed vector field around the solution $\mathbf{s}^{(00)}$,

$$\mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\mathbf{h}^{(ij)}\epsilon^i Z^j, \tag{7.30}$$

where $\mathbf{h}^{(ij)} = \mathbf{h}^{(ij)}(s^{(kl)})$, $(k \le i, l \le i)$. To $O(\epsilon^i)$ we can expand $\mathbf{h}^{(ij)}$ for $j < i$ and we obtain

$$\mathbf{h}^{(ij)} = D\mathbf{f}(\mathbf{s}^{(00)})\mathbf{s}^{(ij)} + \mathbf{G}^{(ij)}, \qquad j = 1, \ldots, i-1, \tag{7.31}$$

where the inhomogeneous term $\mathbf{G}^{(ij)}$ is a function of $\mathbf{s}^{(kl)}$ $(k < i, l < i)$ and $\mathbf{g}(\mathbf{s}^{(00)})$. Similarly, it can be shown that $\mathbf{h}^{(ii)}$ does not depend on $\mathbf{g}$ and only depends on $\mathbf{s}^{(kk)}$, $(k \leq i)$. We can now equate the powers of Z in (7.29) and (7.30) to obtain a set of equations for $\mathbf{s}^{(ij)}$:

$$\dot{\mathbf{s}}^{(ij)} + (j+1)\frac{\mathbf{s}^{(i,j+1)}}{(t-t_*)} = D\mathbf{f}(s^{(00)})\mathbf{s}^{(i0)} + \mathbf{G}^{(ij)}, \qquad j = 0, \dots, i-1, \tag{7.32.a}$$

$$\dot{\mathbf{s}}^{(ii)} = D\mathbf{f}(\mathbf{s}^{(00)})\mathbf{s}^{(ii)} + \mathbf{G}^{(ii)}, \tag{7.32.b}$$

where $\mathbf{G}^{(ii)}$ is a function of $\mathbf{s}^{(kk)}$ $(k = 0, \dots, i-1)$ and not of the perturbation $\mathbf{g}$. The term $\mathbf{G}^{(ij)}$ is polynomial in its arguments which are Laurent series. Therefore, a local expansion can be obtained. Each of the systems in (7.32) can be solved locally. The n arbitrary coefficients of the expansion of $\mathbf{s}^{(i,j+1)}$ can be used to ensure the formal existence of the series $\mathbf{s}^{(ij)}$. By recursion, $\mathbf{s}^{(ii)}$ is first solved. Then, $\mathbf{s}^{(i,i-1)}$ can be found by choosing the n arbitrary constants of the fundamental solution $\mathbf{s}^{(ii)}$ such that no logarithmic term enters in the fundamental solution $\mathbf{s}^{(i,i-1)}$ expressed as a Laurent series around t_*. Similarly, the other solutions can be computed along the same lines. Therefore, if $\mathbf{s}^{(ii)}$ exists for all i, all the others series $\mathbf{s}^{(i,j+1)}$ can be computed recursively. The existence of these series are guaranteed by Theorem 3.7 and are explicitly given in (3.185).

(ii) We assume that all $\mathbf{s}^{(i-1,j)}$ for $j = 1, \dots, i-1$ are convergent series and prove by induction the convergence of $\mathbf{s}^{(ij)}$ for $j = 1, \dots, i$ given by the solution of (7.32). The inhomogeneous term $\mathbf{G}_{ij}$ in (7.32) is polynomial in $\mathbf{s}^{(lk)}$, $(l = 0, \dots, i-1;\ k = 0, \dots, i-1)$ and is therefore convergent in a punctured disk around t_*. According to (7.32), the behavior of $\mathbf{s}^{(ij)}$ at the singularity is

$$\mathbf{s}^{(ij)} \underset{t\to t_*}{\sim} \boldsymbol{\sigma}^{(ij)}(t-t_*)^{\mathbf{p}+\mathbf{p}^{(ij)}-\mathbf{1}}, \tag{7.33}$$

where $\boldsymbol{\sigma}^{(ij)} \in \mathbb{C}^n$ and $\mathbf{p}^{(ij)} \in \mathbb{N}_0^n$ for all i, j. Therefore, from Assumption A6, the behavior of $\mathbf{G}_{ij}$ is

$$\mathbf{G}^{(ij)} \underset{t\to t_*}{\sim} \boldsymbol{\gamma}^{(ij)}(t-t_*)^{\mathbf{p}+\widetilde{\mathbf{p}}^{(ij)}-\mathbf{1}}, \tag{7.34}$$

with $\boldsymbol{\gamma}^{(ij)} \in \mathbb{C}^n$ and $\widetilde{\mathbf{p}}^{(ij)} \in \mathbb{N}^n\ \forall\ i, j$. The change of variables,

$$\mathbf{s}^{(ij)} = (t-t_*)^{\mathbf{p}}\mathbf{v}^{(ij)}, \tag{7.35}$$

transforms system (7.32) into

$$\dot{\mathbf{v}}_{ij} = (t-t_*)^{-1}\left(A^{(ij)}\mathbf{v}^{(ij)} + \mathbf{a}^{(ij)}\right), \qquad j = 1, \dots, i, \tag{7.36}$$

where $A^{(ij)}$, and $\mathbf{a}^{(ij)}$ are, according to Theorem 3.3, convergent Taylor series. Therefore, the singularity t_* is a *regular singular point* and according to Proposition 3.1, the fundamental solutions of (7.36) are convergent Laurent series. □

Using Proposition 7.1, we know that there exists a formal expansion in ϵ, $(t-t_*)$ and $\log(t-t_*)$. However, it is not known, a priori, at which order in ϵ the first logarithmic correction will enter. This is provided by the following proposition.

Proposition 7.2 *Consider the ϵ-expansion of the first non-vanishing compatibility condition $C_\rho = \sum_{i=1}^{\rho} c_{\rho,i}\epsilon^i$ and define d_ϵ to be the lowest degree in ϵ of this polynomial. Then, in the ϵ-expansion of (7.28), we have*

$$\mathbf{s}^{(ij)} = 0\ \forall\ i < d_\epsilon, \qquad j = 1, \dots, i. \tag{7.37}$$

Proof. Since the unperturbed system passes Painlevé test #3, $d_\epsilon > 0$ and the expansions (7.28) can be written

$$\mathbf{x} = \sum_{i=0}^{\infty} \mathbf{x}^{(i)}\epsilon^i, \tag{7.38.a}$$

$$\mathbf{x}_i = \sum_{j=0}^{i-d_\epsilon+1} \mathbf{s}^{(ij)}Z^j. \tag{7.38.b}$$

If $d_\epsilon > 1$, then up to $O(\epsilon^{d_\epsilon - 1})$, the compatibility conditions are satisfied. Therefore, the series $\mathbf{s}^{(ij)}$ ($i = 1, \dots, d_\epsilon - 1;\ j = 1, \dots, i$) can be set to zero and ϵ-expansion (7.28) simplifies to (7.38). □

7.3 The Melnikov vector in n dimensions

The Melnikov vector in n dimensions is the multidimensional analog of the Melnikov integral for planar systems introduced in Section 7.0.1 in that it gives an estimate of the distance between stable and unstable manifolds in different phase space directions. The Melnikov vector also provides a criterion for the existence of a transverse intersection of these manifolds which, under further assumptions, may lead to the existence of chaotic dynamics of the Smale horseshoe type (Wiggins, 1988). Let $W_{\mathrm{u}},\ W_{\mathrm{s}} \in \mathbb{R}^n$ be the unstable and stable manifolds of the hyperbolic fixed point for the unperturbed system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Similarly, let by $W_{\mathrm{u}}^{\epsilon},\ W_{\mathrm{s}}^{\epsilon} \in \mathbb{R}^n \times S^1$ be the stable and unstable manifolds of the perturbed system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, \theta) \qquad \mathbf{x} \in \mathbb{R}^n, \tag{7.39.a}$$

$$\dot{\theta} = 1 \qquad \theta \in S^1. \tag{7.39.b}$$

By assumption, the stable and unstable manifolds intersect on a set of dimension of at least one ($W_{\mathrm{u}} \cap W_{\mathrm{s}} \neq \emptyset$). We want to know if there are transverse homoclinic intersections ($W_{\mathrm{u}}^{\epsilon} \pitchfork W_{\mathrm{s}}^{\epsilon} \neq \emptyset$) for $\epsilon \neq 0$. If this is the case, there exists a bi-infinite set of homoclinic intersections in the perturbed dynamics.

7.3.1 The variational equation

To study the persistence of homoclinic orbits in the perturbed dynamics of (7.39), we examine the variational equation around the homoclinic solution defined by A3 by formally expanding the perturbed solution $\mathbf{x} = \mathbf{x}(t)$ in powers of ϵ:

$$\mathbf{x} = \sum_{i=0}^{\infty} \mathbf{x}^{(i)} \epsilon^i. \tag{7.40}$$

To $O(\epsilon^0)$, the unperturbed system is recovered, that is, $\dot{\mathbf{x}}^{(0)} = \mathbf{f}(\mathbf{x}^{(0)})$, for which the solution of interest is the homoclinic curve $\mathbf{x}^{(0)} = \hat{\mathbf{x}}(t)$. To $O(\epsilon^1)$, we obtain

$$\dot{\mathbf{x}}^{(1)} = D\mathbf{f}(\mathbf{x}^{(0)})\mathbf{x}^{(1)} + \mathbf{g}(\mathbf{x}^{(0)}, t + t_0), \tag{7.41}$$

where we have explicitly introduced the dependence on the initial time t_0. Following Section 5.1, the homogeneous part of (7.41) defines the *variational equation* and the *adjoint variational equation*:

$$\dot{\mathbf{u}} = D\mathbf{u}, \tag{7.42.a}$$

$$\dot{\bar{\mathbf{u}}} = -\bar{\mathbf{u}}D, \tag{7.42.b}$$

where $D = D\mathbf{f}(\hat{\mathbf{x}}(t))$ is the Jacobian matrix evaluated on the homoclinic solution and $\mathbf{u}$ (resp. $\bar{\mathbf{u}}$) is a column (resp. row) vector in $\mathbb{R}^n$. The variables in the adjoint system are denoted by an overbar.

To define the Melnikov vector, we use a particular fundamental solution of the adjoint variational system. We first find a particular fundamental solution which is periodic in the ith-direction, the imaginary period being determined by the Floquet multipliers. It is possible to take advantage of this periodicity to define a Melnikov vector which can be estimated by the method of residues. Indeed, as a result of Assumption A3 on the periodicity of the homoclinic curve in the imaginary direction, it is possible to construct a fundamental solution with the following property (Coddington & Levinson, 1955; Hille, 1969).

Lemma 7.1 *(i) The variational equation (7.42.a) has a fundamental Floquet solution $Q \in \mathrm{GL}(n, \mathbb{C}(t))$ where every column solution $\mathbf{q}^{(j)}$ has the property*

$$\mathbf{q}^{(j)}(t + \omega i) = \frac{1}{s_j}\mathbf{q}^{(j)}(t) \quad \forall\ \mathbf{q}^{(j)} \in Q,\ \forall\ t \in \mathbb{R}. \tag{7.43}$$

(ii) The adjoint variational equation (7.42.b) has a fundamental Floquet solution $\bar{Q} \in \mathrm{GL}(n, \mathbb{C}(t))$ *where every row solution* $\bar{\mathbf{q}}^{(j)}$ *has the property*

$$\bar{\mathbf{q}}^{(j)}(t + \omega i) = s_j \bar{\mathbf{q}}^{(j)}(t) \quad \forall\, \bar{\mathbf{q}}^{(j)} \in \bar{Q},\ \forall\, t \in \mathbb{R}, \tag{7.44}$$

with Floquet multipliers

$$s_j = e^{-i\lambda_j \omega}, \qquad j = 1, \dots, n, \tag{7.45}$$

where $\{\lambda_1, \dots, \lambda_n\}$ *are the eigenvalues of the hyperbolic fixed point. Moreover, we can choose* Q *and* $\bar{Q}$ *such that* $Q\bar{Q} = \mathrm{I}$.

Proof. From Assumption A3, the entries of $D = D(t)$ are $i\omega$-periodic in time $D(t + i\omega) = D(\hat{\mathbf{x}}(t + i\omega)) = D(\hat{\mathbf{x}}(t)) = D(t)$. As a consequence, properties (7.43-7.44) hold. To determine the shift s_i, we note that (7.43) is valid for all time and in particular for $t \to \infty$ where system (7.42.b) becomes

$$\dot{\bar{\mathbf{u}}} = -\bar{\mathbf{u}} D\mathbf{f}(\mathbf{0}). \tag{7.46}$$

From Assumption A2, Equation (7.46) is a linear system with constant coefficients which can be diagonalized. In the variables, $\bar{\mathbf{v}} = C\bar{\mathbf{u}}$, where D is diagonal, it reads

$$\dot{\bar{\mathbf{v}}} = -\bar{\mathbf{v}}\mathrm{diag}(\boldsymbol{\lambda}). \tag{7.47}$$

The solutions of this system provide us with the values of the shift s_i, that is, the Floquet multiplier (7.45). Moreover, by using Proposition 5.1, we can choose the two fundamental Floquet solutions Q and $\bar{Q}$ such that $Q\bar{Q} = \mathrm{I}$. □

To further simplify the problem, we consider differential equations described by Assumption A1. Let $\widetilde{Q} \in \mathcal{M}_{l\times n}(\mathbb{R})$ be the set of bounded solutions of (7.42.b). There exist l' $(0 < l' < n)$ independent bounded solutions. We now show that under Assumption A1, this number l is equal to the number of independent first integrals, that is, $l' = l$.

Lemma 7.2 *Under assumption A1, the Floquet multipliers of all bounded solutions* $\bar{\mathbf{q}}^{(j)} \in \widetilde{Q}$ *of system (7.42.b) are equal to one.*

Proof. Using Proposition 5.2, if $I = I(\mathbf{x})$ is a first integral of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then $\mathbf{w}(t) = \partial_{\mathbf{x}} I(\hat{\mathbf{x}})$ is a solution of its adjoint variational equation. Now, let $\{\mathbf{w}_j = \partial_{\mathbf{x}} I_j(\hat{\mathbf{x}}(t)),\ j = 1, \dots, l\}$ be the set of solutions given by the first integrals. We show that this set provides a basis for the set of bounded solutions of the adjoint and that all of these solutions are associated with identity floquet multipliers. Due to the autonomous character of I_j, we have

$$\mathbf{w}_j(t + \omega i) = \mathbf{w}_j \quad \forall\, t \;\in\; \mathbb{R}. \tag{7.48}$$

The set of bounded solutions of a linear system of differential equations forms a subgroup of the group of solutions. Therefore, any linear superposition of $\mathbf{w}_j$ for $j = 1, \dots, l$ with constant coefficients is a bounded solution with property (7.48). Recall that the number of independent first integrals, l, for an n-dimensional system, is either equal to $n/2$ for Hamiltonian system or $(n - 1)$ for general systems. In the Hamiltonian case, Chow and Yamashita (1992, Prop 9.3, p. 122) proved that the gradients of first integrals form a basis for the bounded solutions of the adjoint variational equation. In the non-Hamiltonian case, $l = (n - 1)$ and the gradients of the first integrals also form a complete set of bounded solutions. Therefore, in both cases, the number of first integrals l is equal to l', the number of independent bounded solutions of the adjoint variational equation. As a consequence, any solution $\bar{\mathbf{q}} \in \widetilde{Q}$ can be obtained by a linear superposition and enjoys property (7.48). □

7.3.2 The Melnikov vector

Different versions of the Melnikov vector have been given in the literature (Gruendler, 1985; Chow & Yamashita, 1992). The Melnikov vector can be easily obtained by considering the Fredholm alternative for expansion (7.40). To first order, if we apply $\widetilde{Q}$ to (7.41) and integrate by parts over the real axis, we obtain

$$\int_{-\infty}^{+\infty} \widetilde{Q}\mathbf{g}(\mathbf{x}^{(0)}, t + t_0)dt = 0. \tag{7.49}$$

This is a necessary condition for the first order equation (7.41) to support bounded solutions under perturbations. More generally, in order to define a quantity measuring the distance between the perturbed and unperturbed curves in phase space, it is possible to specify $\mathbf{x}^{(0)}$ as the homoclinic solution.

Definition 7.1 Let $\hat{\mathbf{x}} = \hat{\mathbf{x}}(t)$ be a homoclinic solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, and $\widetilde{Q}$ be a complete set of bounded solutions of the adjoint variational equation. Then, the *Melnikov vector* $\mathbf{M}(t_0) \in \mathbb{R}^l$ is

$$\mathbf{M}(t_0) = \int_{-\infty}^{+\infty} \widetilde{Q}\mathbf{g}(\mathbf{x}(t), t + t_0)dt. \tag{7.50}$$

The Melnikov vector is defined up to a linear transformation. Different linear combinations of $\widetilde{Q}$ provide different, yet equivalent, vectors from which we can derive the same conditions for the existence of homoclinic intersections in parameter space. The importance of the Melnikov vector lies in the following proposition (Chow & Yamashita, 1992, p. 102).

Theorem 7.1 *Let $\hat{\mathbf{x}} = \hat{\mathbf{x}}(t)$ be a homoclinic solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and let $\mathbf{M}(t_0)$ be its Melnikov vector. Then, under a sufficiently small perturbation ϵ, the stable and unstable manifolds $W_{\mathrm{u}}^{\epsilon}(\mathbf{0})$, $W_{\mathrm{s}}^{\epsilon}(\mathbf{0})$ have a transverse homoclinic intersection if there exists $t_1 \in \mathbb{R}$ such that*

$$\mathbf{M}_i(t_1) = 0, \qquad i = 1, \dots, l, \quad \text{and} \tag{7.51.a}$$

$$\frac{d\mathbf{M}_i}{dt_0}(t_1) \neq 0, \quad \text{for at least one } i. \tag{7.51.b}$$

The Melnikov vector derived in Gruendler (1985) (see also (Wiggins, 1992)) was obtained by geometric methods to show that it is the projection of the distance vector between the stable and unstable manifolds onto the direction $\mathbf{q}^{(i)}(t)$ of the linear part of vector field $\mathbf{g}$ along the homoclinic orbit $\hat{\mathbf{x}}(t)$. In Chow and Yamashita, the Melnikov vector is defined from the condition for the existence of homoclinic orbits in the perturbed dynamics. The Fredholm alternative is used to show the existence of the formal expansion (7.40) (see also Keener (1982) and Broomhead and Rowlands (1982)). The derivation of the Melnikov vector by using the Fredholm alternative is particularly relevant in our context since we use the Fredholm alternative to obtain local compatibility conditions for the existence of a Laurent series. The connection between these two approaches will be given explicitly in the next section.

7.3.3 The method of residues

Due to the aforementioned particular choice of the fundamental solution, it is possible to estimate the Melnikov vector by the method of residues. To do so, we consider the Fourier expansion of $\mathbf{g}(\hat{\mathbf{x}}(t), t + t_0)$ and $\mathbf{M}(t_0)$. That is,

$$\mathbf{g}(\hat{\mathbf{x}}(t), t + t_0) = \sum_{k=-\infty}^{+\infty} \mathbf{g}^{(k)}(\hat{\mathbf{x}})e^{ik(t+t_0)}, \tag{7.52}$$

and

$$\mathbf{M}(t_0) = \sum_{k=-\infty}^{+\infty} \mathbf{M}^{(k)}e^{ikt_0}, \tag{7.53}$$

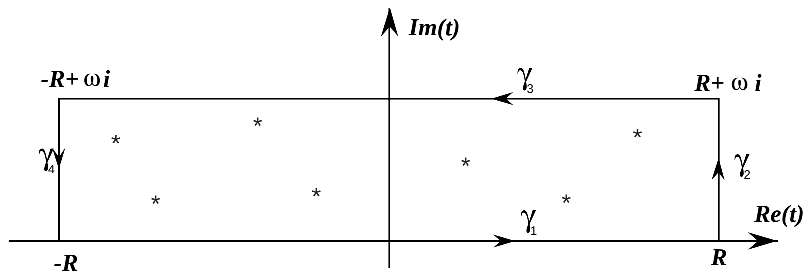

Figure 7.3: The integration contour γ.

where

$$\mathbf{M}^{(k)} = \int_{-\infty}^{+\infty} \widetilde{Q}\mathbf{g}^{(k)}(\hat{\mathbf{x}}(t))e^{ikt}. \tag{7.54}$$

Since every $\mathbf{M}^{(k)}$ can be estimated by the method of residues, we have the following result.

Proposition 7.3 *Under assumptions A1 to A5, the Melnikov vector is*

$$\mathbf{M}(t_0) = \sum_{k=-\infty}^{+\infty} \mathbf{M}^{(k)} e^{ikt_0}, \tag{7.55}$$

with

$$\mathbf{M}^{(0)} = -\frac{2\pi}{\omega N} \sum_{t_* \in S} \operatorname*{res}_{t_*} \Big[\sum_{t_* \in S} \widetilde{Q}\, \mathbf{g}^{(0)}(\hat{\mathbf{x}}(t))(t - t_*) \Big], \tag{7.56}$$

and

$$\mathbf{M}^{(k)} = \frac{2\pi i}{1 - e^{-\omega k}} \sum_{t_* \in S} \operatorname*{res}_{t_*} \Big[\widetilde{Q}\, \mathbf{g}^{(k)}(\hat{\mathbf{x}}) e^{ikt} \Big]. \tag{7.57}$$

Proof. First consider the coefficients of $\mathbf{M}^{(k)}$ with $k \neq 0$ and define for the homoclinic solution $\hat{\mathbf{x}} = \hat{\mathbf{x}}(t)$, the path $\gamma = \bigcup_{i=1}^{4} \gamma_i$ where

$$\gamma_1 = [-R, R], \tag{7.58.a}$$
$$\gamma_2 = [R, R + \omega i], \tag{7.58.b}$$
$$\gamma_3 = [R + \omega i, -R + \omega i], \tag{7.58.c}$$
$$\gamma_4 = [-R + \omega i, -R]. \tag{7.58.d}$$

with R chosen large enough so that all of the singularities of $\hat{\mathbf{x}}(t)$ are included in γ (see Figure 7.3). Let $\bar{\mathbf{q}} \in \widetilde{Q}$. Since the system is assumed to be completely integrable (Assumption A1), the Floquet multiplier of $\bar{\mathbf{q}}$ is one. Now, consider the following contour integral

$$I_{\gamma}^{(k)} = \int_{\gamma} \bar{\mathbf{q}}.\mathbf{g}^{(k)}(\hat{\mathbf{x}}(t))e^{ikt}dt. \tag{7.59}$$

The k-th Fourier component of the Melnikov integral can be expressed as

$$M^{(k)} = \lim_{R\to\infty} I_{\gamma_1}^{(k)}, \tag{7.60}$$

where $M^{(k)}$ is the component of the Melnikov vector corresponding to the solution $\bar{\mathbf{q}}$. The integral in (7.60) can be split into four parts $I_{\gamma}^{(k)} = \sum_{i=1}^{4} I_{\gamma_i}^{(k)}$. Since $\bar{\mathbf{q}}$ is bounded and $\mathbf{g}^{(k)}$ is polynomial in $\hat{\mathbf{x}}(t)$, we obtain,

$$I_{\gamma_2}^{(k)} + I_{\gamma_2}^{(-k)} \underset{R\to\infty}{\to} 0, \tag{7.61.a}$$

$$I_{\gamma_4}^{(k)} + I_{\gamma_4}^{(-k)} \underset{R\to\infty}{\to} 0. \tag{7.61.b}$$

To compute I_{γ_3} in terms of I_{γ_1} we use assumption A3 to obtain

$$I_{\gamma_3}^{(k)} = -e^{-\omega k} I_{\gamma_1}^{(k)}. \tag{7.62}$$

Combining (7.59)-(7.60) and (7.62), we find, by the theorem of residues (Remmert, 1991, p. 381), the component of the Melnikov vector associated with the solution $\bar{\mathbf{q}}$:

$$M(t_0) = M^{(0)} + \sum_{k\neq 0} \frac{2\pi i e^{ikt_0}}{1 - e^{-\omega k}} \sum_{t_*\in S} \operatorname*{res}_{t_*} \Big[\bar{\mathbf{q}}.\mathbf{g}^{(k)}(\hat{\mathbf{x}}(t)) e^{ikt} \Big]. \tag{7.63}$$

That is,

$$\mathbf{M}(t_0) = \mathbf{M}^{(0)} + 2\pi i \sum_{k\neq 0} \frac{e^{ikt_0}}{(1 - e^{-\omega k})} \sum_{t_*\in S} \operatorname*{res}_{t_*} \Big[\widetilde{Q}\, \mathbf{g}^{(k)}(\widehat{x}(t)) e^{ikt} \Big]. \tag{7.64}$$

Second, we evaluate $M^{(0)}$ and notice that for $k \neq 0$, the formulae for $M^{(k)}$ (7.63) is not valid. To compute $M^{(0)}$ in terms of residues, we consider the integral of $\sum_{t_*\in S}(t-t_*)\widetilde{Q}\mathbf{g}^{(0)}(\widehat{x}(t))$ on the contour γ defined in (7.58). That is,

$$\begin{aligned} I_\gamma &= \int_\gamma \sum_{t_*\, \in\, S} \Big[\widetilde{Q}\, \mathbf{g}^{(0)}(\hat{\mathbf{x}}(t))(t-t_*)dt \Big], \qquad (7.65)\\ &= \sum_{t_*\, \in\, S} \Big\{ \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} \Big\} \Big[\widetilde{Q}\, \mathbf{g}^{(0)}(\hat{\mathbf{x}}(t))(t-t_*)dt \Big]. \end{aligned}$$

Here again, we use the boundedness of $\bar{\mathbf{q}}$ and $\hat{\mathbf{x}}(t)$ to evaluate the border contributions:

$$I_{\gamma_2} + I_{\gamma_4} \underset{R\to\infty}{\to} 0. \tag{7.66}$$

Now, consider the contribution on γ_3:

$$\begin{aligned} &\sum_{t_*\, \in\, S} \int_{R+\omega i}^{-R+\omega i} \widetilde{Q}\, \mathbf{g}^{(0)}(\hat{\mathbf{x}}(t))(t-t_*)dt = \\ &\qquad \int_{R}^{-R} \widetilde{Q}\, \mathbf{g}^{(0)}(\hat{\mathbf{x}}(t))(t-t_*)dt - \omega i \int_{-R}^{R} \widetilde{Q}\, \mathbf{g}^{(0)}(\hat{\mathbf{x}}(t))dt, \qquad (7.67) \end{aligned}$$

where we have used the change of variable $t \to t - \omega i$. The second term in the right hand side of (7.67) is proportional to the Melnikov vector. Taking into account the two last relations and using again the residues of the integral I_γ, we obtain

$$\begin{aligned} \lim_{R\to\infty} I_\gamma &= -N\omega i \mathbf{M}^{(0)}, \\ &= 2\pi i \sum_{t_*\, \in\, S} \operatorname*{res}_{t_*} \Big[\sum_{t_*\, \in\, S} \widetilde{Q}\, \mathbf{g}^{(0)}(\hat{\mathbf{x}}(t))(t-t_*) \Big]. \qquad (7.68) \end{aligned}$$

The appearance of the number of poles N in the this last equation is due to the fact that in (7.65) we introduced both a contour integral, which included all the poles and a sum over all the singularities. Hence, $M^{(0)}$ is found to be

$$\mathbf{M}^{(0)} = -\frac{2\pi}{\omega N} \sum_{t_* \in S} \operatorname*{res}_{t_*} \left[\sum_{t_* \in S} \widetilde{Q}\, \mathbf{g}^{(0)}(\hat{\mathbf{x}}(t))(t-t_*) \right]. \tag{7.69}$$

□

As a consequence, both $\mathbf{M}^{(0)}$ and $\mathbf{M}^{(k)}$ can be obtained from local analysis. We now evaluate these residues by the singularity analysis.

7.4 Singularity Analysis and the Melnikov vector

If Assumptions A1-A5 hold, the Melnikov vector can be estimated by the method of residues. Under Assumptions A1 and A6, the Ψ-ϵ-expansion can be applied and formal solutions can be found in the complex plane. Therefore, under Assumptions A1-A6, an explicit form for the residues can be found by considering the ϵ-expansions. Indeed, locally around the singularity t_*, it is possible to find and use the fundamental solutions of the variational equations to compute the residues of the Melnikov integrand.

7.4.1 The fundamental local solution

First, we consider the variational equation (7.42.a) and its adjoint (7.42.b):

$$\dot{\mathbf{u}} = D\mathbf{u}, \tag{7.70.a}$$
$$\dot{\bar{\mathbf{u}}} = -\bar{\mathbf{u}}D, \tag{7.70.b}$$

where $\mathbf{u}, \bar{\mathbf{u}} \in \mathbb{C}^n$ and $D = D\mathbf{f}(\hat{\mathbf{x}})$. These two systems have local fundamental solutions given by the following proposition.

Proposition 7.4 *(i) System (7.70.a) has a fundamental local solution $Y \in \mathrm{GL}(n, \mathbb{C}(t))$ such that every column $\mathbf{y}^{(i)} \in Y$ has a convergent local expansion in a punctured disk $B(t_*, \mathbf{y}^{(i)})$ around t_* given by*

$$\mathbf{y}^{(i)}(t) = (t-t_*)^{\mathbf{p}+\rho_i} \left(\boldsymbol{\beta}^{(i)} + \sum_{j=1}^{\infty} \mathbf{b}_j^{(i)} (t-t_*)^j \right). \tag{7.71}$$

(ii) The adjoint system (7.70.b) has a fundamental local solution $\bar{Y} \in \mathrm{GL}(n, \mathbb{C}(t))$ such that every row $\bar{\mathbf{y}}^{(i)} \in \bar{Y}$ has a convergent local expansion in a punctured disk $B(t_*, \bar{\mathbf{y}}^{(i)})$ around t_* given by

$$\bar{\mathbf{y}}^{(i)}(t) = (t-t_*)^{-\mathbf{p}-\rho_i} \left(\bar{\boldsymbol{\beta}}^{(i)} + \sum_{j=0}^{\infty} \bar{\mathbf{b}}_j^{(i)} (t-t_*)^j \right), \tag{7.72}$$

where $\mathbf{p} \in \mathbb{Z}^n$ is the dominant behavior of the unperturbed system (Assumption A4), ρ_i is a Kovalevskaya exponent, $\boldsymbol{\beta}^{(i)}$ and $\bar{\boldsymbol{\beta}}^{(i)}$ are eigenvectors of K and K^T and $\mathbf{b}_j^{(i)}, \bar{\mathbf{b}}_j^{(i)} \in \mathbb{C}^n$.

Proof. We prove *(i)*. Part *(ii)* can be proved in the same way. We first construct the solutions whose convergence follows from Proposition 7.1. Let $\mathbf{y}(t)$ be any solution of (7.70.a). The dominant behavior of $\mathbf{y}(t)$ around the pole t_* is given by $\mathbf{y} = \beta(t-t_*)^{\mathbf{q}}$ for some $\mathbf{q} \in \mathbb{Z}^n$. Inserting $\mathbf{y} = \boldsymbol{\beta}(t-t_*)^{\mathbf{q}}$ in (7.70.a) and taking the limit for $t \to t_*$, we find

$$(t-t_*)^{q_i - \mathbf{1}} \boldsymbol{\beta}_i \mathbf{q}_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\boldsymbol{\alpha}) \boldsymbol{\beta}_i (t-t_*)^{q_j + p_i - p_j - 1} \tag{7.73}$$

$$= \sum_{j=1}^{n} (K + \mathrm{diag}(\mathbf{p}))_{ij} \boldsymbol{\beta}_j (t-t_*)^{q_j + p_i - p_j - 1}. \tag{7.74}$$

From these relations, the leading exponent of $\mathbf{y}$ and the vector $\boldsymbol{\beta}$ can be estimated as $q_i = p_i + \rho$ where ρ is an eigenvalue of K of eigenvector $\boldsymbol{\beta}$. The formal existence of series (7.71) is ensured by Assumption A4. This analysis shows that $t = t_*$ is a regular singular point for system (7.70.a). Hence, according to Proposition 3.1, there exist n convergent linearly independent solutions. □

The fundamental local solutions Y, and $\bar{Y}$ enjoy the same property as the Floquet solutions Q and $\bar{Q}$.

Lemma 7.3 *The two fundamental solutions Y, $\bar{Y}$ are such that in $B(t_*, Y) \cap B(t_*, \bar{Y})$, we have*

$$Y\bar{Y} = \bar{Y}Y = \mathrm{I}, \tag{7.75}$$

where $B(t_*, Y) = \cap_{i=1}^{n} B(t_*, \mathbf{y}^{(i)})$ and $B(t_*, \bar{Y}) = \cap_{i=1}^{n} B(t_*, \bar{\mathbf{y}}^{(i)})$.

Proof. The Floquet solutions Q and $\bar{Q}$ are such that $\dot{Q} = DQ$ and $\dot{\bar{Q}} = -\bar{Q}D$. That is, around a singularity t_*, we have

$$(t - t_*)\dot{Q} = \widetilde{D}Q, \tag{7.76.a}$$

$$(t - t_*)\dot{\bar{Q}} = -\bar{Q}\widetilde{D}, \tag{7.76.b}$$

where $\widetilde{D} = D\mathbf{f}((t - t_*)^{-\mathbf{p}}\mathbf{x}^{(0)}(t - t_*))$ and $\mathbf{x}^{(0)}$ is the Laurent expansion of the homoclinic solution around t_*. Note that, $\widetilde{D}$ is such that $\widetilde{D}(0) = K$. We introduce the linear transformations $Q = CY$ and $\bar{Q} = \bar{Y}C^{-1}$, where $C \in \mathrm{GL}(n, \mathbb{C})$ is chosen such that $C^{-1}KC = \mathrm{diag}(\boldsymbol{\rho})$. The new fundamental solutions Y and $\bar{Y}$ are precisely the ones defined in Proposition 7.4. Therefore, using Lemma 5.1, we find

$$\bar{Q}Q = \bar{Y}Y = \mathrm{I}, \tag{7.77.a}$$

$$C^{-1}Q\bar{Q}C = Y\bar{Y} = \mathrm{I}, \tag{7.77.b}$$

from which the result follows. □

7.4.2 Residues and local solutions

We showed that it is possible to construct local solutions of the perturbed system (7.14) to $O(\epsilon^1)$ in terms of an ϵ-expansion of the form

$$\mathbf{x} = \mathbf{x}^{(0)} + \epsilon\mathbf{x}^{1)} + O(\epsilon^2), \tag{7.78.a}$$

$$\mathbf{x}^{(1)} = \mathbf{s}^{(10)} + \mathbf{s}^{(11)}Z, \tag{7.78.b}$$

where $Z = \log(t - t_*)$ and $\mathbf{s}^{(10)}$, $\mathbf{s}^{(11)}$ are convergent Laurent series around t_*. A necessary and sufficient condition to have a solution $\mathbf{x}^{(1)}$ of the form (7.78.b) with non-vanishing $\mathbf{s}^{(11)}$ is provided by Proposition 7.2. That is, there exists at least one Kovalevskaya exponent ρ_i such that

$$c_{r_i,1} \neq 0. \tag{7.79}$$

in the compatibility conditions C_{r_i} (see (7.26)). It is now possible to give an explicit form for the solution $\mathbf{x}^{(1)}$ in terms of the local fundamental solutions $Y, \bar{Y}$.

Proposition 7.5 *To first order in ϵ, the solution $\mathbf{x}$ of system (7.14) under Assumptions A1 to A6 is given by (7.78.a)-(7.78.b) where*

$$\mathbf{s}^{(10)} = Y\mathbf{K}^{(0)}(t), \tag{7.80.a}$$

$$\mathbf{s}^{(11)} = Y\mathbf{K}^{(1)}, \tag{7.80.b}$$

with

$$\mathbf{K}^{(0)}(t) = \int^t \bar{Y}(s)\mathbf{g}(\hat{\mathbf{x}}(s), s)ds - \mathbf{K}^{(1)}\log(t - t_*), \tag{7.81.a}$$

$$\mathbf{K}^{(1)} = \operatorname*{res}_{t_*}\Big[\bar{Y}(t)\mathbf{g}(\hat{\mathbf{x}}(t), t)\Big]. \tag{7.81.b}$$

Proof. Equations for $\mathbf{s}^{(10)}$ and $\mathbf{s}^{(11)}$ are easily derived by inserting (7.78.b) in

$$\dot{\mathbf{x}}^{(1)} = D\mathbf{x}^{(1)} + \mathbf{g}(\mathbf{x}^{(0)}, t), \tag{7.82}$$

and by considering each order in Z:

$$\dot{\mathbf{s}}^{(10)} = D\mathbf{s}^{(10)} - \mathbf{s}^{(11)}(t - t_*)^{-1} + \mathbf{g}(\mathbf{x}^{(0)}, t), \tag{7.83.a}$$

$$\dot{\mathbf{s}}^{(11)} = D\mathbf{s}^{(11)}. \tag{7.83.b}$$

The general solution of this system is given by $\mathbf{s}^{(1i)} = Y\mathbf{K}^{(1)}$, for $i = 0, 1$. The method of variation of constants gives

$$\mathbf{K}^{(0)}(t) = \int^t \bar{Y}(s)\mathbf{g}(\hat{\mathbf{x}}(s), s)ds - \mathbf{K}^{(1)}\log(t - t_*). \tag{7.84}$$

It is now possible to choose $\mathbf{K}^{(1)}$ such that $\mathbf{s}^{(10)}$ is a Laurent solution. That is,

$$\mathbf{K}^{(1)}(t_*) = \operatorname*{res}_{t_*}\Big[\bar{Y}(t)\mathbf{g}(\hat{\mathbf{x}}(t), t)\Big]. \tag{7.85}$$

□

We have reached a crucial stage in our construction. On the one hand, we have shown that the Melnikov vector can be computed by the residues. On the other hand we have shown that the residues are determined by the Ψ-ϵ expansion. Now, if we compare expression (7.64) with (7.85) we see that vector $\mathbf{K}^{(1)}(t_*)$ is needed to compute the Melnikov vector $\mathbf{M}(t_0)$. By construction, $\mathbf{K}^{(1)}(t_*)$ is given by the coefficients $O(t - t_*)^{p+r_i}$ of the series $\mathbf{s}^{(11)}$, and due to Assumption A5, its first component vanishes identically. Furthermore, $\mathbf{s}^{(11)}$ can be found by direct inspection of the ϵ-expansion of the Ψ-series (7.27),

$$\mathbf{x} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \mathbf{a}^{(ijk)}(t - t_*)^{i+\mathbf{p}}\epsilon^j Z^k. \tag{7.86}$$

Therefore, vector $\mathbf{K}_1(t_*)$ is given by

$$K_1^{(1)}(t_*) = 0, \tag{7.87.a}$$

$$K_i^{(1)}(t_*) = \bar{\boldsymbol{\beta}}^{(i)}.\mathbf{a}^{(\rho_i,11)}, \qquad i = 2, \ldots, n. \tag{7.87.b}$$

7.4.3 The compatibility conditions and the residues

The last step of our construction is to relate the residues of the Melnikov vector to the compatibility conditions of the system. To do so, we use a convenient form of vector $\mathbf{K}^{(1)}$ is given by the different compatibility conditions C_{ρ_i} defined by (7.24) for the nonintegrable system (7.14). To show this explicitly, we consider a formal Laurent expansion with a logarithmic correction for the perturbed system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, t)$ up to the first positive Kovalevskaya exponent ρ:

$$\mathbf{x} = \sum_{i=0}^{\rho} \mathbf{a}_i(t - t_*)^{\mathbf{p}+i} + \mathbf{b}(t - t_*)^{\mathbf{p}+\rho}\log(t - t_*)$$
$$+ O((t - t_*)^{\mathbf{p}+\rho+1}, \log(t - t_*)^2). \tag{7.88}$$

Note that $\frac{\partial \mathbf{b}}{\partial \epsilon}(\epsilon = 0)$ can be identified with the first non-vanishing coefficient of $\mathbf{s}^{(11)}$, namely, $\mathbf{a}_\rho$. Using recursion relation (7.23) for the $\mathbf{a}_i$'s, we can compute all of the coefficients up to $\mathbf{a}_\rho$,

$$K\mathbf{a}_\rho = \rho\mathbf{a}_\rho - \mathbf{P}_\rho(\mathbf{a}_0, \ldots, \mathbf{a}_{\rho-1}) + \mathbf{b}, \tag{7.89}$$

where $\mathbf{P}_\rho$ is polynomial. Applying the eigenvector $\bar{\boldsymbol{\beta}}$ of K^T of eigenvalue ρ to both sides of this relation, we obtain

$$C_r \equiv \bar{\boldsymbol{\beta}}.\mathbf{P}_\rho = \bar{\boldsymbol{\beta}}.\mathbf{b}. \tag{7.90}$$

Using the definition of the compatibility conditions in (7.24) and the relation $\frac{\partial \mathbf{b}}{\partial \epsilon}(\epsilon = 0) = \mathbf{a}_\rho$, we find that the arbitrary constant K_{12} defined in (7.87.a) is related to the conditions in (7.90) to first order, namely,

$$K_{12} = \frac{\partial C_\rho}{\partial \epsilon}(\epsilon = 0). \tag{7.91}$$

This process is repeated for the other Kovalevskaya exponents up to ρ_n and we obtain

$$K_{1i}(t_*) = \bar{\boldsymbol{\beta}}^{(i)}.\mathbf{s}^{(11)}_{\rho_i} = c_{\rho_i}, \qquad i = 2, \dots, n, \tag{7.92}$$

where $c_{\rho_i} = \frac{\partial C_{\rho_i}}{\partial \epsilon}(\epsilon = 0)$. The compatibility condition C_{ρ_i} is obtained by the usual singularity analysis procedure as described in Chapter 3. The dependence of the compatibility conditions to the position of the singularities is rather subtle. The positions t_* of the singularities do not appear explicitly in the coefficients of the Laurent expansion for a perturbation $\mathbf{g}^{(k)}(\hat{\mathbf{x}}(t))$. But around different singularities, the leading order behavior of the Laurent series will be different. For instance, the homoclinic solution $\hat{\mathbf{x}}(t)$ for the Duffing system (7.1) has two different leading behaviors at singularities $t_*^{(1)}$ and $t_*^{(2)}$. In this case, we find that $K_{\rho_i}(t_*^{(1)}) = -K_{\rho_i}(t_*^{(2)})$. Now, we expand K_1 and c_{ρ_i} in Fourier series. That is,

$$K_1 = \sum_{k=-\infty}^{+\infty} K_1^{(k)} e^{ikt_0}, \tag{7.93.a}$$

$$K_1^{(k)} = \operatorname*{res}_{t_*}\left[\bar{Y}\mathbf{g}^{(k)}(\hat{\mathbf{x}}(t))e^{ikt}\right]. \tag{7.93.b}$$

The compatibility condition C_{ρ_i} is a periodic function of t_* of period 2π. Therefore, c_{r_i} can be expanded in Laurent series and reads

$$c_{\rho_j} = \sum_{k=-\infty}^{+\infty} c^{(k)}_{\rho_j}(t)e^{ikt_0}, \quad j = 2, \dots, n. \tag{7.94}$$

Proposition 7.6 *Assume that Assumptions A1 to A6 hold. Then, for $k \neq 0$, $\mathbf{M}^{(k)}$ is given by*

$$M_j^{(k)} = \frac{2\pi i}{\left(1 - e^{-k\omega}\right)} \sum_{t_* \in S} c^{(k)}_{\rho_j}(t_*)e^{ikt_*}, \quad j = 2, \dots, n. \tag{7.95}$$

Proof. On the one hand, we proved that the residues of $\bar{Y}\mathbf{g}^{(k)}(\hat{\mathbf{x}}(t))e^{ikt}$ can be expressed as a function of the compatibility conditions. On the other hand, we showed that the Melnikov vector is a sum of the residues of $\bar{Q}\mathbf{g}^{(k)}(\hat{\mathbf{x}}(t))e^{ikt}$. Since $\bar{Q}$ and $\bar{Y}$ are two fundamental solutions of the adjoint variational equation, locally, there exists an invertible linear transformation C mapping one solution to the other, $\bar{Q} = \bar{Y}C^{-1}$, and we can use the local form of Y to compute the Melnikov vector by the method of residues.

□

The number of components of the Melnikov vector defined in (7.95) is actually greater than the number l of independent first integrals of the system. However, since we are only interested in the non-vanishing components of the Melnikov vector it is defined by the l non-vanishing components of (7.95) and some compatibility conditions vanish identically. For example. the condition associated with $\rho = -1$.

7.5 The algorithmic procedure

7.5.1 The computation of the Melnikov vector: a non-algorithmic procedure

The Melnikov vector can be computed directly from expression (7.50), as prescribed in Gruendler (1985). This procedure includes the following steps.

1. Find a closed analytic form for the homoclinic curve $\hat{\mathbf{x}} = \hat{\mathbf{x}}(t)$ which connects the fixed point ($\mathbf{x} = \mathbf{0}$) to itself.

2. Compute a closed-form for the fundamental matrix solution $\bar{Q}$. This involves the computation of a solution of a linear system of ODEs with time-dependent coefficients.

3. Using both the bounded subset $\widetilde{Q}$ of the fundamental solution and the homoclinic solution for $\mathbf{g}(\hat{\mathbf{x}})$, compute the Melnikov vector (7.50) which is given by l improper integrals over the real axis.

7.5.2 The algorithmic procedure

The Melnikov vector can be computed from expression (7.95) in the following three steps.

1. **The homoclinic solution.** Show the existence of a homoclinic solution which satisfies Assumption A1-A6. Find ω, the period in the imaginary direction and the position of the singularities t_* for the homoclinic solution. This can be achieved either analytically when enough information on the homoclinic manifold can be gathered or numerically using integration techniques in the complex plane such as ATOMCC (Chang & Corliss, 1980). Even though, the homoclinic solution does not have to be computed, its main properties should be known. The local analysis around the hyperbolic fixed point determines the linearized stable and unstable manifolds $W_{\mathrm{u}}, W_{\mathrm{s}}$. Let $\lambda_{\mathrm{s}}, \lambda_{\mathrm{u}}$ be the linear eigenvalues along the stable and unstable eigendirections of a one-dimensional homoclinic solution. Then, the i-periodicity ω can be obtained by considering the ratio

$$\delta = -\frac{\lambda_{\mathrm{u}}}{\lambda_{\mathrm{s}}}. \tag{7.96}$$

If $\delta \notin \mathbb{Q}$, the homoclinic solution is not i-periodic. If $\delta \in \mathbb{Q}$, the homoclinic solution is periodic with i-period given by

$$\omega = \frac{2\pi}{|\mathrm{Re}(\lambda)|}. \tag{7.97}$$

where λ is the least common multiplier of $\{\lambda_{\mathrm{u}}, \lambda_{\mathrm{s}}\}$.

2. **The Ψ-ϵ series.** Find the first term of the Ψ-ϵ expansion of the solution $\mathbf{x} = \mathbf{x}^{(0)} + \epsilon\mathbf{x}^{(1)}$ up its highest Kovalevskaya exponent. To do this, compute the coefficients $\mathbf{a}_i^{(jk)}$ of the series $\mathbf{s}^{(jk)}$ appearing in expansion (7.8) for $i = 0, \dots, \rho_n$, $j = 0, 1$ and $k = 0, 1$. That is,

$$\begin{aligned}\mathbf{x} = (t - t_*)^{\mathbf{p}} \sum_{i=0}^{\rho_n} \Bigg(& \mathbf{a}_i^{(00)} (t - t_*)^i \\ & + \epsilon \Big(\mathbf{a}_i^{(10)} (t - t_*)^i + \mathbf{a}_i^{(11)} Z (t - t_*)^i \Big) \Bigg) \\ & + O(\epsilon^2). \end{aligned} \tag{7.98}$$

where $Z = \log(t-t_*)$. Otherwise, compute the compatibility conditions for the perturbed system. These conditions can be found by inserting the truncated expansion,

$$\mathbf{x} = \sum_{i=0}^{\rho_{max}} \mathbf{a}_i (t-t_*)^{\mathbf{p}+i}, \tag{7.99}$$

into the perturbed system where ρ_{max} is the last Kovalevskaya exponent. This expansion is not a solution of the system, but the compatibility conditions C_{ρ_i} for the existence of a solution of this type can be used to compute the ith component of the Melnikov vector.

3. **The Melnikov vector.** The last step consists of computing the different Fourier components of the Melnikov vector $\mathbf{M}^{(k)}$ from the compatibility conditions (see Equations (7.95)). The Melnikov vector is then given by

$$M_j(t_0) = M_j^{(0)} + 2\pi i \sum_{k=-\infty}^{+\infty} \frac{e^{ikt_0}}{1-e^{-k\omega}} \sum_{t_*\in S} c_{\rho_j}^{(k)}(t_*) e^{ikt_*}, \quad j = 1, \dots, l. \tag{7.100}$$

The component $\mathbf{M}^{(0)}$ can also be computed by singularity analysis as illustrated in the examples given below.

7.6 Some illustrative examples

In order to illustrate and comment on our method, we choose three well-known examples which illustrate different aspects of the method.

Example 7.2 The ABC flow. The ABC flow is a model of three-dimensional steady flow. Although, it has a simple Euler representation it can produce a chaotic Lagrangian structure (Dombre *et al.*, 1986; Zhao *et al.*, 1993). In the Cartesian space coordinates (x, y, z) with 2π-boundary conditions, the equations for the flow are

$$\dot{x} = A\sin z + C\cos y, \tag{7.101.a}$$
$$\dot{y} = B\sin x + A\cos z, \tag{7.101.b}$$
$$\dot{z} = C\sin y + B\cos x. \tag{7.101.c}$$

Here, we assume that $A > B > 0$, $A, B \in \mathbb{R}$ and $C = \epsilon \ll 1$. For $C = 0$, the system is completely integrable with a first integral $I \equiv B\sin x + A\cos z = h$, where h is constant along each orbit and $y = ht$ with a proper choice of initial conditions. In order to study this system, we write it as a polynomial vector field and apply, for the nearly-integrable case, the change of variables $(q_1, p_1, q_2, p_2) = (B\cos x, B\sin x, A\cos z, A\sin z)$ and $y = ht$ to obtain

$$\dot{q}_1 = -p_1(p_2 + \epsilon\cos(ht)), \tag{7.102.a}$$
$$\dot{p}_1 = q_1(p_2 + \epsilon\cos(ht)), \tag{7.102.b}$$
$$\dot{q}_2 = -p_2(q_1 + \epsilon\sin(ht)), \tag{7.102.c}$$
$$\dot{p}_2 = q_2(q_1 + \epsilon\sin(ht)). \tag{7.102.d}$$

Step 1. The homoclinic solution. The variable transformation maps the heteroclinic separatrices to homoclinic curves. For $\epsilon = 0$, the system has a homoclinic solution whose limit set is the hyperbolic fixed point $(q_1, p_1, q_2, p_2) = (0, -B, A, 0)$. That is,

$$q_1 = \frac{-2ab\sinh(at)}{b^2+\cosh^2(at)}, \quad p_1 = B\frac{b^2+1-\sinh^2(at)}{b^2+\cosh^2(at)}, \tag{7.103.a}$$

$$q_2 = A\frac{-b^2+\cosh^2(at)}{b^2+\cosh^2(at)}, \quad p_2 = A\frac{2b\cosh(at)}{b^2+\cosh^2(at)}, \tag{7.103.b}$$

where $a^2 = AB$ and $b^2 = \frac{B}{A-B}$. The homoclinic solution is periodic in the imaginary direction with period $\omega = \frac{2\pi}{a}$ and has four singularities in the fundamental domain given by

$$t_*^{(1,2)} = \frac{1}{2}\left(i\pi \pm \log\left(\frac{c+1}{c-1}\right)\right), \tag{7.104.a}$$

$$t_*^{(3,4)} = \frac{1}{2}\left(3i\pi \pm \log\left(\frac{c+1}{c-1}\right)\right), \tag{7.104.b}$$

with $c^2 = A/B$.

Step 2. The singularity analysis. The unperturbed system, written in the variables $\mathbf{x}_0 = (q_1, p_1, q_2, p_2)$, has four different Laurent expansions around the movable singularities

$$\mathbf{x}^{(0)} = \boldsymbol{\alpha}_i (t - t_*)^{-1} \left(1 + \sum_{j=1}^{\infty} \mathbf{a}_j (t - t_*)^j\right), \tag{7.105}$$

for which the Kovalevskaya exponents are $\rho \in \{-1, 1, 2, 2\}$ and

$$\boldsymbol{\alpha}_1 = (-i, 1, -1, i), \quad \boldsymbol{\alpha}_2 = (i, -1, 1, -i), \tag{7.106.a}$$

$$\boldsymbol{\alpha}_3 = (-i, -1, 1, i), \quad \boldsymbol{\alpha}_4 = (i, 1, -1, i). \tag{7.106.b}$$

Inserting (7.105) into the perturbed system (7.102.a) we find that no Laurent expansion exists for the solution. Nevertheless, the compatibility condition at the Kovalevskaya exponent $\rho = 1$ provides us with the data to compute the residues of the Melnikov integral. That is,

$$K(t_*^{(1)}) = \frac{i}{\sqrt{2}}(\cos(ht_*^{(1)}) + \sin(ht_*^{(1)})), \tag{7.107.a}$$

$$K(t_*^{(2)}) = \frac{-i}{\sqrt{2}}(\cos(ht_*^{(2)}) - \sin(ht_*^{(2)})), \tag{7.107.b}$$

$$K(t_*^{(3)}) = \frac{-i}{\sqrt{2}}(\cos(ht_*^{(3)}) + \sin(ht_*^{(3)})), \tag{7.107.c}$$

$$K(t_*^{(4)}) = \frac{i}{\sqrt{2}}(\cos(ht_*^{(4)}) - \sin(ht_*^{(4)})), \tag{7.107.d}$$

where $K(t_*) = \frac{dC(t_*)}{d\epsilon}(\epsilon = 0)$ is the compatibility condition for $\rho = 1$ to $O(\epsilon)$.

The compatibility condition for $\rho = 2$ vanishes identically to $O(\epsilon)$ and thus no contribution to the Melnikov vector will enter at this order. The Kovalevskaya exponent $\rho = 2$ corresponds to the first integrals of degree two

$$q_1^2 + p_1^2 = B^2, \quad q_2^2 + p_2^2 = A^2, \tag{7.108}$$

which are preserved under the perturbation. Hence, the Melnikov components corresponding to these first integrals vanish identically.

Step 3. The Melnikov vector. We compute the sum of the residues for each Fourier component and obtain

$$\begin{aligned} K^{(\pm 1)} &= \sum_{i=1}^{4} K^{(\pm 1)}(t_*^{(i)}), \\ &= \frac{1}{\sqrt{2}} e^{\mp 2\gamma}(e^{\gamma} - e^{-\gamma})(\cos\xi - \sin\xi), \end{aligned} \tag{7.109}$$

with $\gamma = \frac{h\pi}{2a}$, $\xi = \frac{h}{2a}\log\left(\frac{c+1}{c-1}\right)$. The Melnikov vector has a unique non-vanishing component:

$$\begin{aligned} M(t_0) &= 2\pi i\left[\frac{K^{(+1)}e^{it_0}}{(1-e^{-4\gamma})} + \frac{K^{(-1)}e^{-it_0}}{(1-e^{4\gamma})}\right] \\ &= \sqrt{2}\pi(\cos\xi - \sin\xi)\text{sech}\gamma\sin t_0. \end{aligned} \tag{7.110}$$

We conclude from the expression of the Melnikov integral that, for ϵ small enough (*i.e.*, C small enough), the system has infinitely many transversal homoclinic intersections and, as a consequence, has chaotic streamlines of the Smale horseshoe type. ■

Example 7.3 A two-degree-of-freedom pendulum

We consider an undamped magnetized spherical pendulum(Gruendler, 1985):

$$\dot{p}_1 = q_1, \tag{7.111.a}$$
$$\dot{q}_1 = q_1 - 2q_1(q_1^2 + q_2^2) + \epsilon G_1, \tag{7.111.b}$$
$$\dot{p}_2 = q_2, \tag{7.111.c}$$
$$\dot{q}_2 = q_2 - 2q_2(q_1^2 + q_2^2) + \epsilon G_2, \tag{7.111.d}$$

where the perturbation is given by

$$G_1 = 3\mu_1 q_1 + \mu_2 p_1 + F(t)\Bigg(\mu_3(3q_1^2 + q_2^2) + 2\mu_4 q_1 q_2\Bigg), \tag{7.112.a}$$

$$G_2 = \mu_1 q_2 + \mu_2 p_2 + F(t)\Bigg(2\mu_3 q_1 q_2 + \mu_4(q_1^2 + 3q_2^2)\Bigg). \tag{7.112.b}$$

We assume that the Taylor expansion of the periodic function F ($F(t+T) = F(t)$) is known up to the third term ($F(t) = \sum_{i=0}^{3} f_i(t-t_*)^i + O((t-t_*)^4)$).

Step 1. The homoclinic solution. Due to the radial symmetry of the unperturbed system, there exists a family of homoclinic curves $\hat{\mathbf{x}}(t;\theta) = (q_1, q_2) = \text{sech}(t)(\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi]$ whose periodicity in the imaginary direction is $\omega = 2\pi$ and whose singularities are located at $t_*^{(1)} = i\pi/2$ and $t_*^{(2)} = 3i\pi/2$. Assumptions A1-A6 are satisfied and the computation of the Melnikov vector reduces to the computation of the compatibility conditions for the nonintegrable system.

Step 2. Evaluation of $\mathbf{M}^{(k)}$ by singularity analysis. We first compute the residues from the singularity analysis of the nonintegrable system. The compatibility conditions for the perturbed system are found by expanding the solutions in terms of Laurent series up to the highest Kovalevskaya exponent $q_1 = \sum_{i=0}^{4} a_i(t-t_*)^{i-1}$ and $q_2 = \sum_{i=0}^{4} b_i(t-t_*)^{i-1}$. These truncated expansions are not solutions of the perturbed system. Instead they are a convenient way to obtain the compatibility conditions for the nonintegrable system and they contain the residues of the Melnikov integral around a movable singularity of the unperturbed system. The Kovalevskaya exponents for the unperturbed systems are $\rho \in \{-1, 0, 3, 4\}$. Inserting the expansions for q_1 and q_2 into (7.111), and balancing terms to each power of $(t-t_*)$, we find conditions on the parameters for the existence of the coefficients a_3 and b_3 and (a_4, b_4). Let C_{r_3}, C_{r_4} be such conditions. According to our algorithm, the residues K_i of the Melnikov vector around the singularity t_* is given by

$$\begin{aligned} K_3(t_*) &= i\frac{(2f_2 - f_0)}{2}(\mu_3\sin\theta - \mu_4\cos\theta), \\ &= i\frac{(1+\nu^2)}{2}\cos(\nu t_*)(\mu_3\sin\theta - \mu_4\cos\theta), \end{aligned} \tag{7.113.a}$$

$$\begin{aligned} K_4(t_*) &= \frac{(6f_3 - f_1)}{2}(\mu_3\cos\theta + \mu_4\sin\theta), \\ &= -i\nu\frac{(1+\nu^2)}{2}\sin(\nu t_*)(\mu_3\cos\theta + \mu_4\sin\theta), \end{aligned} \tag{7.113.b}$$

where $K_i(t_*) = \frac{dC_{r_i}}{d\epsilon}(\epsilon = 0)$ and $F(t) = \cos(\nu t)$.

Step 2bis. Computation of $\mathbf{M}^{(0)}$. According to (7.69), the computation of $\mathbf{M}^{(0)}$ reduces to the computation of $\bar{Q}\,\mathbf{g}^{(0)}(\hat{\mathbf{x}}(t))(t-t_*)$. This quantity can be evaluated if we note that the residue of $\bar{Q}\,\mathbf{g}^{(0)}(\hat{\mathbf{x}})$ is given by the compatibility conditions of the Laurent series. Therefore, by replacing the perturbation $\mathbf{g}^{(0)}$ by $(t-t_*)\mathbf{g}^{(0)}$, the residue of $(\bar{Q}\,\mathbf{g}^{(0)}(\hat{\mathbf{x}}(t))(t-t_*))$ is given by the compatibility conditions for the system under the new perturbation. Therefore, from (7.69), we have

$$M_3^{(0)} = -2\mu_1 \sin(2\theta), \tag{7.114.a}$$

$$M_4^{(0)} = \frac{2}{3}\mu_2. \tag{7.114.b}$$

Step 3. The Melnikov vector. Taking into account the two different contributions of the Melnikov vector, we obtain

$$M_3 = -2\mu_1 \sin(2\theta) + \frac{(1+\nu^2)}{2}\pi \mathrm{sech}(\pi/2)(\mu_3 \sin\theta - \mu_4 \cos\theta)\cos(\nu t_0),$$

$$M_4 = \frac{2}{3}\mu_2 - \nu\frac{(1+\nu^2)}{2}\pi \mathrm{sech}(\pi/2)(\mu_3 \cos\theta + \mu_4 \sin\theta)\sin(\nu t_0).$$

From the analysis of the Melnikov vector, it is possible to obtain the conditions for the homoclinic intersections in parameter space and therefore, transversal or tangential intersections can be predicted (Chow & Yamashita, 1992). ■

Example 7.4 The Hénon-Heiles Hamiltonian. The Hénon-Heiles system has many interesting features. First, it is Hamiltonian and exhibits chaotic behavior for varying energies. Second, it can be integrated partially or completely according to the parameter values using singularity analysis as shown in Section 3.9.1 (Hénon & Heiles, 1964; Tabor, 1989). The Melnikov analysis has been applied in detail by different authors (Holmes, 1982). The system can be written

$$\dot{q}_1 = p_1 \tag{7.115.a}$$

$$\dot{p}_1 = -\nu^2 q_1 + \sqrt{2} q_1^2 + \epsilon G(q_1 - q_2), \tag{7.115.b}$$

$$\dot{q}_2 = p_2, \tag{7.115.c}$$

$$\dot{p}_2 = -\nu^2 q_2 - \sqrt{2} q_2^2 - \epsilon G(q_1 - q_2), \tag{7.115.d}$$

where $G(v) = \alpha v + \beta v^2$. The unperturbed system is integrable with two first integrals given by the Hamiltonian for the decoupled subsystems (S_1, S_2)

$$H_1 = \frac{p_1^2 + \nu^2 q_1^2}{2} - \frac{\sqrt{2}}{3} q_1^3, \tag{7.116.a}$$

$$H_2 = \frac{p_2^2 + \nu^2 q_2^2}{2} + \frac{\sqrt{2}}{3} q_2^3. \tag{7.116.b}$$

This system introduces two new features of our analysis. First, Assumption A6 is not satisfied due to the fact that the quadratic term in G is dominant at the singularity. It detunes the Kovalevskaya exponents and introduces irrational Kovalevskaya exponents, which are incompatible with integrability. Even in the case of dominant perturbations, a perturbative Ψ-ϵ expansion can still be obtained even though the convergence of the Laurent series cannot be guaranteed. To first order in ϵ, the Ψ-ϵ expansion includes logarithmic corrections from which we can extract the relevant data for the Melnikov vector. Second, the forcing is not external. It is due to the coupling between the subsystems $S1$ and $S2$ by choosing initial conditions such that $S1$ evolves on the homoclinic solution and $S2$ has periodic solutions. As a consequence, $S2$ affects the system $S1$ via the coupling G like an external forcing.

Step 1. The homoclinic solution. The two subsystems ($S1$,$S2$) are Painlevé integrable and support two different expansions. The first one is a Laurent series with dominant behavior at the singularities $\mathbf{p} = (-2, -3)$ and Kovalevskaya exponents $\rho \in \{-1, 6\}$. The second one is a Taylor expansion. The two systems have a homoclinic orbit whose limit sets are the hyperbolic fixed points $(q_1, p_1)=(\nu^2/\sqrt{2}, 0)$ and $(q_2, p_2)=(-\nu^2/\sqrt{2}, 0)$. The periodicity of the homoclinic solution in the imaginary direction is given by $\omega = \frac{2\pi}{\nu}$. This can be readily obtained by considering the linear eigenvalue around the hyperbolic fixed point $\lambda_\pm = \pm\nu$, $\omega = \frac{2\pi}{|\lambda_\pm|}$ as mentioned in Section 7.1.2. The only singularity of the homoclinic solution in its fundamental domain is located on the imaginary axis at $t_* = \frac{i\pi}{\nu}$. Inside these orbits, there is an infinite sequence of periodic orbits around the center located at the origin. As a consequence, Assumptions A1-A4 are satisfied.

Step 2. The Ψ-ϵ expansion. It is not possible to find a logarithmic correction when considering the compatibility conditions for the existence of a Laurent expansion for the nonintegrable system in complex time. Indeed, there is no Ψ-expansion which includes logarithmic terms for systems with branch points involving irrational powers of $(t - t_*)$. Nevertheless, the Ψ-ϵ expansion can still be obtained. To show this explicitly, we build a formal expansion for system $S1$ under a time-periodic perturbation $q_2 = f(t)$. The expansion in ϵ reads

$$\mathbf{x} = \sum_{i=0}^{\infty} \mathbf{x}^{(n)} \epsilon^n, \tag{7.117}$$

where $\mathbf{x}^{(n)} = (q_1^{(n)}, p_1^{(n)})$. First, we consider the Laurent solutions for the unperturbed system $S1$ (Equation (7.115) with $\epsilon = 0$). That is,

$$\begin{pmatrix} q_1^{(0)} \\ p_1^{(0)} \end{pmatrix} = \frac{\sqrt{2}}{(t-t_*)^2} \tag{7.118}$$
$$\begin{pmatrix} 3 + \frac{\nu}{4}(t-t_*)^2 + \frac{\nu^2}{80}(t-t_*)^4 + a6(t-t_*)^6 + \sum_{j=7}^{\infty} a_j (t-t_*)^j \\ \frac{-6}{(t-t_*)} + \frac{\nu^2}{40}(t-t_*)^3 + 4a_6(t-t_*)^5 + \sum_{j=7}^{\infty}(j-2)a_j(t-t_*)^{j-1} \end{pmatrix},$$

where a_6 is arbitrary (Kovalevskaya exponent $\rho = 6$). To order ϵ, the equation for the first perturbation $\mathbf{x}^{(1)}$ is:

$$\dot{q}_1^{(1)} = p_1^{(1)}, \tag{7.119.a}$$
$$\dot{p}_1^{(1)} = -\nu^2 q_1^{(1)} + \sqrt{2} q_1^{(0)} q_1^{(0)} + \epsilon G(q_1^{(0)} - f(t)). \tag{7.119.b}$$

There exists a formal expansion for $(q_1^{(1)}, p_1^{(1)})$ of the form $\mathbf{x}^{(1)} = \mathbf{s}^{(10)} + \mathbf{s}^{(11)} \log(t - t_*)$, where the $\mathbf{s}^{(ij)}$'s are Laurent series:

$$\mathbf{s}^{(10)} = \sum_{i=0}^{\infty} \mathbf{b}^{(i)} (t-t_*)^{i-3}, \tag{7.120.a}$$
$$\mathbf{s}^{(11)} = \sum_{i=0}^{\infty} \mathbf{c}^{(i)} (t-t_*)^{i+4}, \tag{7.120.b}$$

with $\mathbf{b}^{(i)}, \mathbf{c}^{(i)} \in \mathbb{C}^2$ for all i and where $\mathbf{b}^{(0)}$ and $\mathbf{b}^{(7)}$ are arbitrary coefficients corresponding, respectively, to the Kovalevskaya exponents $\rho = -1$ and $\rho = 6$. The coefficients $\mathbf{b}^{(i)}$ can be computed recursively by inserting (7.120) into (7.119) up to $\mathbf{b}^{(7)}$. At this stage, a compatibility condition enters for the existence of the expansion (7.120.a) and is a function of the arbitrary constant K_0 ($\mathbf{c}^{(0)} = K_0(1, 4)$). The compatibility condition reads

$$K_0(t_*) = \frac{1}{7}\left[\alpha f_2 + \beta\big(f_1^2 + \frac{\omega}{\sqrt{2}} f_2 + 2 f_0 f_2 - 6\sqrt{2} f_4\big)\right], \tag{7.121}$$

where $f(t) = \sum_{i=0}^{\infty} f_i (t-t_*)^i$. The coefficient K_0 is the residue of the Melnikov integrand around the singularity of the homoclinic loop.

Step 3. The Melnikov vector. We choose as a first approximation of the periodic function $q_2 = f(t)$, a solution of the linearized subsystem $S2$ around its fixed point $(q_2, p_2) = (0, 0)$, namely,

$$f(t) = \frac{h}{\nu} \cos \nu t, \tag{7.122}$$

which is valid for h small. Using the Taylor expansion of $f(t)$ around the singularity t_*, the coefficient K_0 simplifies to

$$K_0(t_*) = -\frac{h\nu}{14} \big[\alpha + \frac{\beta\nu}{\sqrt{2}}(\nu - 1)\big] \cos(i\pi) + O(h^2). \tag{7.123}$$

The Melnikov vector, computed from (7.95), is

$$\begin{aligned} M(t_0) &= -2\pi i \frac{h\nu}{14} \big(\alpha + \frac{\beta\nu}{\sqrt{2}}(\nu - 1)\big) \left[\frac{e^{i(i\pi + \nu t_0)}}{1 - e^{-2\pi}} + \frac{e^{-i(i\pi + \nu t_0)}}{1 - e^{2\pi}} \right] + O(h^2), \\ &= \frac{h\pi\nu}{14} \left[\alpha + \frac{\beta\nu}{\sqrt{2}}(\nu - 1) \right] \frac{\sin \nu t_0}{\sinh \pi} + O(h^2). \end{aligned} \tag{7.124}$$

For small h, $M(t_0)$ has simple zeroes and there are homoclinic intersections in the dynamics of the coupled systems. The same computation can be performed for arbitrary periodic functions $f(t)$ once their expansions are known up to fourth order. ■

7.7 Exercises

7.1 Consider the one-and-a-half-degree-of-freedom Hamiltonian (Bountis *et al.*, 1987)

$$H = y(x - x^2) - \epsilon x \cos(t), \tag{7.125}$$

and its associated Hamilton's equation

$$\dot{x} = x - x^2, \tag{7.126.a}$$
$$\dot{y} = -y + 2xy + \epsilon \cos t. \tag{7.126.b}$$

(i) Show that this system has, for $\epsilon = 0$, two hyperbolic fixed points and a heteroclinic orbit $\hat{\mathbf{x}}(t) = \left(\frac{1}{1+e^t}, 0\right)$. *(ii)* Compute the Melnikov integral by the method of residues and show that invariant manifold intersect transversally. *(iii)* Plot the perturbed orbits in the $x - y$ plane. *(iv)* Compute the ϵ expansion of the Ψ-series. *(v)* Adapt the argument of Section 7.4 to the case of a heteroclinic orbit and compute the Melnikov integral from the Ψ-series. *(vi)* Show that the perturbed dynamics does not have a Smale horseshoe since the stable and unstable manifolds of the fixed points extend to infinity and that the system does not exhibit chaos.

7.2 Consider the perturbed planar system (Ramani *et al.*, 1984; Bountis *et al.*, 1987)

$$\dot{x} = x - x^2 + 3xy, \tag{7.127.a}$$
$$\dot{y} = -y - y^2 + 3xy + \epsilon\left(\gamma \cos \omega t + \delta x\right). \tag{7.127.b}$$

First, consider the unperturbed system for $\epsilon = 0$. *(i)* Show that it passes Painlevé test #2 and prove that it has the Painlevé property. *(ii)* Show that the system has a *heteroclinic cycle* constructed from the heteroclinic orbits connecting the fixed points $(0, 0)$ to $(1, 0)$ and $(0, -1)$. *(iii)* Find the explicit form of the three heteroclinic orbits and show that they are periodic in the imaginary direction. Compare the imaginary period of the heteroclinic orbit with the linear eigenvalues at the fixed points.
Second, consider the perturbed case when $\epsilon \neq 0$. *(iv)* Compute the ϵ-expansion for the Ψ-series. *(v)* Use the method of residues to compute the splitting of the heteroclinic orbit connecting $(1, 0)$ to $(0, -1)$. *(vi)* Show that the residues of the Melnikov integrand can be computed from the Ψ-series.

7.3 The perturbed Lorenz equations

$$\dot{x} = y - \epsilon\alpha\left(1 - m \sin(\Omega t)\right) x, \tag{7.128.a}$$
$$\dot{y} = x(1 - z) - \epsilon\beta y, \tag{7.128.b}$$
$$\dot{z} = xy - \epsilon z, \tag{7.128.c}$$

was proposed as a model for a parametrically driven CO_2 laser with modulated losses (Bruhn, 1991). *(i)* Show that for $m = 0$, this system reduces to the Lorenz system (1.35) in the form given by Robbins (1979). *(ii)* Find the three fixed points of the system for $m = 0$ and compute the linear eigenvalues. *(iii)* Show that for small values of ϵ, the origin is a hyperbolic point with a two-dimensional stable manifold. *(iv)* Show that for $\epsilon = 0$, the reduced system is completely integrable with $I_1 = y^2 + (z - 1)^2$ and $I_2 = x^2 + y^2 + (z - 2)^2$. *(v)* Compute the intersection of the level sets $I_1 = C_1^2$ and $I_2 = C_2^2$ and find conditions on C_1 and C_2 for the existence of a pair of homoclinic orbits to the origin. *(vi)* Verify that

$$x = \pm 2C_1 \text{sech}(C_1 t), \tag{7.129.a}$$
$$y = \mp 2C_1^2 \text{sech}(C_1 t)\tanh(C_1 t), \tag{7.129.b}$$
$$z = 1 - C_1^2 + 2C_1^2 \text{sech}^2(C_1 t), \tag{7.129.c}$$

is a homoclinic solution for the reduced system. *(vii)* For ϵ small and $m \neq 0$ apply the procedure of Section 7.5.2 to compute the Melnikov vector and identify the values of the parameters that lead to transverse homoclinic intersections.

Chapter 8:

Open problems

"Who cares about integrability?"
Segur (1991).

As a conclusion, I would like to outline some open problems in the theory of integrability for dynamical systems. I suspect that some of these problems might already have found an answer or are in the process of being answered.

1. **Bounds on the degree of first integrals.** One of the main problem in the computation of first integrals is the absence of a bound on the degree of a possible first integral or a Darboux polynomial. For planar vector field, a theoretical bound on the degree of first integral exists but it cannot be computed explicitly (Singer, 1992). The existence of such a bound is not clear for vector field of dimension $n > 2$ but the natural relationship with the Kovalevskaya exponents given by Theorem 5.6 suggests that if such a bound exists, it is related to the largest Kovalevskaya exponent of a hyperbolic balance. Furthermore, I conjecture that if the weight-homogeneous component of highest weight of a given vector field is completely integrable with polynomial first integrals $I_1, \ldots, I_k$, the degree of a first integral for the complete system cannot be larger than the highest degree of the first integrals $I_1, \ldots, I_k$. This result can be easily proved for planar vector field and could probably be generalized to higher dimensions.
2. **First integrals and geometry.** Consider an n-dimensional vector field and assume that for particular values of the parameters sufficiently many first integrals are known to determine the dynamics. How can we use this information for other values of the parameters? Giacomini and Neukirch (1997; 2000) have shown on particular examples that the first integrals can be used in the nonintegrable regimes to build generalized Lyapunov functions and obtain bounds on the chaotic attractors of three-dimensional vector fields and prove the absence of homoclinic orbits. Their analysis relies on the complete integrability of the highest weight-homogeneous component of the vector field. Can we generalize their approach and provide a general algorithm for the construction of both Lyapunov functions and generalized Lyapunov functions based on the integrability of some components of the vector field?

3. **Lax pairs and singularity analysis.** Lax pairs are recognized as the ultimate proof of integrability. However, to date, there is no general algorithmic procedure to compute them. If has been repeatedly shown that singularity analysis can be used to build Lax pairs for integrable PDEs and many disparate results are available (Weiss, 1983; Tabor & Gibbon, 1986; Newell *et al.*, 1987). However, the construction of these Lax pairs relies on *ad hoc* methods individually tailored for each example and, surprisingly, the method does not seem to be applicable to systems of ODEs. I believe that the problem lies in the fact that PDEs in one dependent variable have only a few dominant balances and a general expansion in Laurent series around a singularity manifold captures the behavior of the general solution. Simple PDEs such as KdV have a unique dominant balance and the general solution can be captured by a unique local expansion. When PDEs have two dominant balances with equal dominant behavior, a pair of local expansions can still be used to build the Lax pairs (Estévez & Gordoa, 1998; Estévez, 1999). Again, the procedure is by no means straightforward or algorithmic. Systems of ODEs typically have multiple balances with different dominant

behaviors and the usual tricks do not seem to be applicable. A better understanding of the relationship between the structure of the local solutions and the Lax pairs may give us a clue on how to build Lax pairs.

4. **A nonlinear Kovalevskaya-Yoshida theory.** The analysis of the local solutions around the singularity was done using the companion systems. We showed that these solutions can be obtained by a normal form analysis of the corresponding companion systems. It would be of great theoretical interest to bypass the construction of the companion system and to develop a normal form theory of local solutions around their singularities directly in terms of the original variables. A theory of this type has already been proposed for the perturbed Euler equations in Costin (1997).

5. **Geometry of blow-up orbits.** We have established necessary conditions for the existence of blow-up manifolds. However, the geometry of the flows on these manifolds has not been investigated and there are many intriguing possibilities. For instance, we could have orbits blowing up both backward and forward in finite time. These orbits would be the equivalent of homoclinic and/or heteroclinic orbits for regular dynamical systems and could allow us to describe the geometry of basins of attractions of finite time singularities.

6. **Distribution of singularities for irregular flows.** The pattern of complex time singularities of integrable systems is believed to be regular. What can we say about the pattern of singularities for nonintegrable system? We saw that the addition of logarithmic terms in the series implies, in most cases, that the singularities tend to cluster on self-similar curves (or at least their projection do). There is, to date, no understanding of the pattern of singularities and their fractal nature observed in many nonintegrable systems. Tabor and Weiss (1981) and Frisch and Morf (1981) show that the dynamical behavior known as "intermittency" is related the distance of the singularities to the real axis and Bountis (1992) found that singularities tend to cluster on chimney patterns. The analysis of nearly-integrable dynamical systems can probably reveal the change in singularity position for relevant orbit and for small perturbation.

7. **Ziglin's theory for dynamical systems.** Ziglin's theory relies heavily on the underlying symplectic structure of Hamiltonian systems. Can we formulate a general Ziglin's theory or Picard-Vessiot theory for non-Hamiltonian systems?

8. **Nonintegrable dynamics.** Consider a two-degree-of-freedom Hamiltonian, H, and assume that it is nonintegrable in the sense of Ziglin, that is, there is no additional analytic first integral. Clearly, this does not imply that the system exhibit chaos in the real phase space. We could, for instance, build an Hamiltonian whose dynamics is unbounded for all initial conditions and, hence, preventing it from exhibiting chaotic dynamics. However, nonintegrability carries over the complex domain and the system might be chaotic on the complexified phase space. Does a nonintegrable Hamiltonian system always exhibit chaos on the complexified phase space?

9. **Exponentially small splitting of singularities.** When an autonomous system of ordinary differential equations possessing a homoclinic or heteroclinic separatrix is perturbed by a rapidly oscillating forcing term, the resulting separatrix splitting becomes exponentially small in the perturbation parameter– with the result that any first order approximation technique for measuring this splitting (*e.g.* the Melnikov method) apparently loses its validity since discarded terms of second order and higher could be larger than the exponentially small first order term (Holmes *et al.*, 1988). The problem of computing such a distance was first outlined by Poincaré (Poincaré, 1890) and has been shown to appear in a number of generic problems in nonlinear dynamics such as the averaging of perturbation (Sanders, 1982), the discretization of continuous vector fields (Fiedler & Scheurle, 1996), the divergence of Poincaré-Birkhoff normal forms (Arnold, 1988a) and the splitting of heteroclinic cycles in KAM theory (Rudnev & Wiggins, 1998). The general assumptions of the Melnikov theory are not satisfied in the case of exponentially small splitting (namely,

Glossary

$\mathbb{K}$	A field of constant
$\mathbb{Z}$	The set of integers
$\mathbb{N}$	The set of positive integers
$\mathbb{Q}$	The rational numbers
$\mathbb{R}$	The real numbers
$\mathbb{C}$	The complex numbers
$\mathbb{K}_0$	$\mathbb{K}_0 = \mathbb{K} \setminus \{0\}$
$\mathbb{K}^+$	The set of positive elements in $\mathbb{K}$ (whenever defined)
$\mathbb{K}_0^+$	The set of strictly positive elements in $\mathbb{K}$ (if defined)
$\mathbb{K}[\mathbf{x}]$	The set of polynomials in $\mathbf{x}$ with coefficients in $\mathbb{K}$
$\mathbb{K}(\mathbf{x})$	The set of rational functions in $\mathbf{x}$ with coefficients in $\mathbb{K}$
$\mathbb{K}((t))$	The set of convergent power series in t with coefficients in $\mathbb{K}$
$\mathbb{K}[[t]]$	The set of formal power series in t with coefficients in $\mathbb{K}$
C^k	The set of functions of class C^k (k times differentiable)
$\mathrm{M}_{m\times n}(\mathbb{K})$	The set of $m \times n$ matrices over $\mathbb{K}$
$\mathrm{M}_n(\mathbb{K})$	The set of $n \times n$ matrices over $\mathbb{K}$
$\mathrm{GL}(n, \mathbb{K})$	The set of invertible $n\times n$ matrices over $\mathbb{K}$
$\mathrm{SL}(n, \mathbb{K})$	The set of $n\times n$ matrices over $\mathbb{K}$ with determinant ± 1

The sets

In the following table, $\mathbb{K}$ is a field of constant, $\mathbf{a}, \mathbf{b}$ are two vectors in $\mathbb{K}^n$, $A \in \mathrm{M}_n(\mathbb{K})$ and $B \in \mathrm{M}_{m\times n}(\mathbb{K})$ are two matrices over $\mathbb{K}$.

$\mathrm{Spec}(A)$	The spectrum of A (the set of eigenvalues)
$\mathrm{Ker}(A)$	The kernel of A (*i.e.*, $\{\mathbf{a} \in \mathbb{K}^n \| A\mathbf{a} = \mathbf{0}\}$)
$\mathrm{diag}(\mathbf{a})$	The diagonal matrix with diagonal elements equal to a_i
I_n	The $n \times n$ identity matrix
I	The identity matrix (whenever the dimension is obvious)
$\mathbf{ab}$	$(\mathbf{ab})_i = a_i b_i,\ i = 1, \ldots, n$ (no summation)
$\mathbf{a.b}$	$\mathbf{a.b} = \sum_{i=1}^{n} a_i b_i^*$
$\|\mathbf{a}\|$	The norm of $\mathbf{a}$ ($\|\mathbf{a}\| = \sqrt{\mathbf{a.a}}$)
$\mathbf{A}_{.i}$	$\mathbf{A}_{.i}$ is the i-th vector column
$\mathbf{A}_{i.}$	$\mathbf{A}_{i.}$ is the i-th row vector
$A \oplus \mathbf{a}$	$(A \oplus \mathbf{a})_{ij} = A_{ij} + a_j$
$\mathbf{a}^{\mathbf{b}}$	$\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^{n} a_i^{b_i}$

Vectors and matrices operations

$S(A, B; \mathbf{x}, t)$	A quasimonomial (QM) system
$\mathcal{Q}_n$	The set of QM systems in n dimensions
$\mathcal{P}_n$	The polynomial systems in n dimensions
$T_C(\mathbf{x})$, $(C \in \mathrm{GL}(n, \mathbb{C})$	A quasimonomial transformation (QMT)
$T_C(S(A, B; \mathbf{x}, t)) = S(C^{-1}A, BC; \mathbf{x}', t)$	A QMT acting on a QM system
$N_{\boldsymbol{\beta}}(dt) = \mathbf{x}^{\boldsymbol{\beta}} dt$	A new-time transformation (NTT)
$N_{\boldsymbol{\beta}}(S(A, B; \mathbf{x}, t)) = S(A, B \oplus \boldsymbol{\beta}; \mathbf{x}, \widetilde{t})$	A NTT acting on a QM system

The quasimonomial formalism

$\mathbf{x} = (t - t_*)^{\mathbf{p}} \sum \mathbf{a}_i (t - t_*)^i$	A Laurent series ($\mathbf{p} \in \mathbb{Z}$) around t_*
$\mathbf{x} = \boldsymbol{\alpha}(t - t_*)^{\mathbf{p}}$	A similarity invariant solution
$\mathbf{p}$	The dominant exponents
$q^{(i)}$ $(q^{(i)} > {}^{(j)} > 0, i > j)$	The non-dominant exponents
$\mathbf{f} = \sum_{i=1}^{l} \mathbf{f}^{(i)}$	A weight-homogeneous decomposition of $\mathbf{f}$
$\mathcal{F} = \{\boldsymbol{\alpha}, \mathbf{p}\}$	A dominant balance
$\tilde{\mathcal{F}} = \{\{\boldsymbol{\alpha}^{(i)}, \mathbf{p}^{(i)}\}, i = 1, \ldots, k\}$	The set of all dominant balances
$K = D\mathbf{f}^{(0)}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p})$	The Kovalevskaya matrix
$\rho \in \mathrm{Spec}(K)$	A Kovalevskaya exponent
$\mathcal{R} = \{-1, \rho_1, \ldots, \rho_n\}$	The set of all Kovalevskaya exponents
$\boldsymbol{\rho} = (1, \rho_1, \ldots, \rho_n)$	The vector all Kovalevskaya exponents

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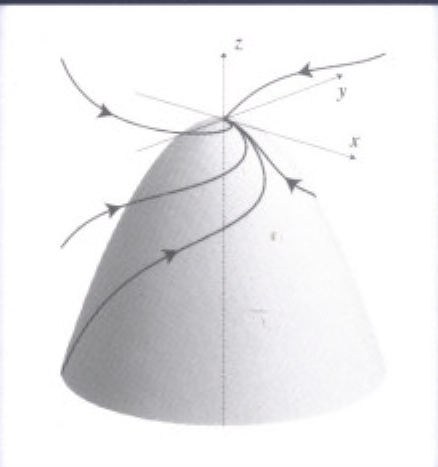
INTEGRABILITY AND NONINTEGRABILITY OF DYNAMICAL SYSTEMS



This invaluable book examines qualitative and quantitative methods for nonlinear differential equations, as well as integrability and nonintegrability theory. Starting from the idea of a constant of motion for simple systems of differential equations, it investigates the essence of integrability, its geometrical relevance and dynamical consequences. Integrability theory is approached from different perspectives, first in terms of differential algebra, then in terms of complex time singularities and finally from the viewpoint of phase geometry (for both Hamiltonian and non-Hamiltonian systems). As generic systems of differential equations cannot be exactly solved, the book reviews the different notions of nonintegrability and shows how to prove the nonexistence of exact solutions and/or a constant of motion. Finally, nonintegrability theory is linked to dynamical systems theory by showing how the property of complete integrability, partial integrability or nonintegrability can be related to regular and irregular dynamics in phase space.

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